A Twistorial Interpretation of the Weierstrass Representation Formulae.

by

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Contents.

Introduction.

The theory of minimal surfaces represents an important chapter in the study of global analysis and remains a testing ground for our understanding of the non-linear partial differential equations of geometry. Perhaps its greatest charm lies in its mercurial avoidance of isolation. Today we see profound applications to such diverse fields as 3-manifold topology and non-abelian gauge theory, to name two ; see [E&L] for a recent survey and extensive bibliography. (Very recently there have been exciting new applications of the theory of periodic minimal surfaces in $\mathbb{R}^3$ to crystallography, see [T&A&H&H].) Consequently, the principal aim of this thesis, which is to establish the groundwork for the investigation of new interactions between minimal surface theory in $\mathbb{R}^3$, algebraic geometry and soliton theory, see §4.F, is very much in the traditional spirit of the subject.

The oldest and strongest links are with complex function theory ; these were forged when it was realised that a minimal surface in $\mathbb{R}^3$ is, at least locally, the real part of a null holomorphic curve in C^3, and consequently that the Gauss map of such a surface is a holomorphic object, see §1.B and §1.C. This fact is embodied in the classical Weierstrass representation formulae (§1.C), which have been of great utility in the study of minimal surfaces in euclidean space, e.g. see [Os1] and [L]. It was observed by Weierstrass in 1866, that if one locally reparameterises a minimal surface by its Gauss map then these equations become (at least locally) integrable, and hence one obtains algebraic formulae for the surface (§1.C, §3.D), in terms of an auxiliary holomorphic function on the complex projective line, see [W] and [E]. This fact does not feature very prominently in the current minimal surface literature.

In his paper [H2] of 1982 on SU(2)-monopoles, Hitchin provides an algebraic geometric interpretation of these formulae of Weierstrass, which involves twistor transforming the null curve in C^3 into an algebraic curve on the space of affine null planes, Π, (a holomorphic line bundle over $\mathbb{P}_1$). This observation of Hitchin's, was my motivation for this work and §§2,3 consist of a detailed account of the mechanics of this twistor correspondence. The following picture may be helpful here. View a null holomorphic curve in C^3 as a conformal field on a Riemann surface. The essential feature of the twistor transform of 2.8 is that the nullity condition, which acts as an infinitesimal constraint on the conformal fields, is encoded into a boundary condition 'at infinity' for the transformed curve, via the Chern class of Π. The transformed field is locally free, and local reparameterization by the Gauss map linearises the moduli space. This boundary condition at infinity, determines the intersection behaviour of the global holomorphic sections of Π, and thus modulates the infinitesimal structure of the 'trajectory field' obtained by osculating a curve on Π, hence the nullity condition, see 2.15.

Having removed the nullity constraint it becomes possible to conformally compactify the range, see §4.B and §4.C. Thus we are likely to obtain a good picture of moduli spaces of null curves in C^3. An interesting point which emerges from this is that we seem to obtain better moduli spaces if we include all critical points of energy, not just the absolute minima, i.e. we allow branching, see [Os3].

§1 consists mostly of well-known results viewed from a slightly new angle, together with some trivial adjustments of classical results to allow for branch points. We make these adjustments in order to obtain a global picture of the algebraic minimal surfaces of §4 and in particular to obtain an understanding of the structure of minimal surfaces which arise from SU(2)-monopoles, see §4.F. Note that §§2,3 and 4 are prefaced by remarks of an introductory nature.

§1 Preliminaries.

Those readers unfamiliar with the ideas described here should consult [Os 1], see also [G&H] and [L].

(A) Branched Minimal Surfaces in $\mathbb{R}^3$.

Let M denote a Riemann surface and suppose that g, a smooth metric, lies in the conformal equivalence class determined by the complex structure. For any $\xi_0 \varepsilon M$ we can write

$$g_\xi = 2\lambda(\xi)|d\xi|^2 \quad ,$$

where ξ is a local holomorphic coordinate on a neighbourhood, U, of ξ_0, $|d\xi|^2 = \operatorname{Re}(d\xi \otimes d\bar{\xi})$ and $\lambda \varepsilon C^\infty(U,\mathbb{R})$ is strictly positive :

Consequently, Δ_g, the Laplace operator of g, takes the following form on U :

$$\Delta_g = \frac{2}{\lambda}\frac{d}{d\bar{\xi}}\frac{d}{d\xi} \quad .$$

Hence the notion of a harmonic function on M, i.e. $f \varepsilon C^\infty(M,\mathbb{R})$ satisfying $\Delta_g f = 0$, is conformally invariant and therefore well-defined with respect to the complex structure of M.

<u>Definition 1.1</u> (i) A map $\phi : M \longrightarrow \mathbb{R}^3$, is said to be <u>harmonic</u>, if each component is a harmonic function on M.

(ii) A smooth map $\phi : M \longrightarrow \mathbb{R}^3$, is said to be weakly conformal if ϕ is conformal on $M \setminus \{$ critical points of ϕ $\}$.

(iii) A smooth, positive semi-definite global section, g, of the second symmetric power of T^*M, the cotangent bundle of M, which is positive-definite off a discrete set of zeros, will be referred to as a semi-metric on M.

Remark 1.2 It is straightforward to check that the distance function, δ_g, induced on M by a semi-metric g, given by

$$\delta_g(x,y) = \inf_{\gamma \varepsilon X} \int_\gamma ds_g \quad ,$$

where ds_g denotes the line element of g, and

$$X = \{ \text{ piece-wise smooth curves from } x \text{ to } y \} ;$$

is a metric on M.

The set of critical points of a non-constant, weakly conformal harmonic map $\phi : M \longrightarrow \mathbb{R}^3$, is either empty or consists of isolated points. This follows because ϕ is harmonic iff $\frac{d}{d\bar{\xi}} \frac{d}{d\xi} \phi = 0$, i.e. $\frac{d\phi}{d\xi}$ is holomorphic : but since

$$d\phi(\xi_0) = 0 \quad \text{iff} \quad \frac{d\phi}{d\xi}(\xi_0) = 0 \quad ,$$

such a ξ_0 must be isolated.

Let <,> denote the Euclidean structure on $\mathbb{R}^3$ and let ds_ϕ denote the semi-metric induced on M by ϕ ; i.e.

$$ds^2_\phi = \phi^*\langle\ ,\ \rangle = 2\left|\frac{d\phi}{d\xi}\,d\xi\right|^2 .$$

Remark 1.3 (i) ds^2_ϕ is the Riemannian semi-metric induced on M by the Hermitian semi-metric $2\left|\frac{d\phi}{d\xi}\right|^2 d\xi \otimes d\bar{\xi}$.

(ii) Since ds^2_ϕ vanishes at a critical point of ϕ, the Gaussian curvature K_ϕ, of ds^2_ϕ , may be infinite there.

Definition 1.4 (i) A critical point of a non-constant, weakly conformal harmonic map is called a branch point .

We denote the set of branch points of ϕ, by B_ϕ.

(ii) A non-constant, (weakly) conformal harmonic map $\phi : M \longrightarrow \mathbb{R}^3$, will be referred to as a (branched) minimal immersion of M into $\mathbb{R}^3$, and the image is a (branched) minimal surface in $\mathbb{R}^3$.

We close this section with a couple of elementary observations regarding the geometry of a branched minimal surface in $\mathbb{R}^3$. Firstly on an infinitesimal level, observe that since the mean curvature of ds^2_ϕ is proportional to $|\Delta_\phi \phi|$, which is zero , K_ϕ is non-positive on M. Secondly on a global level, recall that it follows from the maximum principle of Hopf, that if $\phi : M \longrightarrow \mathbb{R}^3$ is a branched minimal immersion then M is non-compact. However the branched minimal surfacesof central interest to us here, (i.e. algebraic minimal surfaces), are easily shown to be complete in the following sense of Hopf and Rinow, see [H & R] and [Os 1] :

Definition 1.5 (i) A path $p : [0,1) \to M$ is divergent if for any compact subset $X \subset M$ there exists $t_0 < 1$ such that $p(t) \notin X$ for $t > t_0$.

(ii) A semi-metric g on M is complete, if every divergent path on M has infinite length with respect to ds_g.

(iii) A semi-metric g on M is (1,1) if in every local holomorphic coordinate ξ on M, with domain U say, g has the form $g_\xi = \lambda(\xi)|d\xi|^2$, for some $\lambda \in C^\infty(U,\mathbb{R})$.

Proposition 1.6 A (1,1) semi-metric g, on a Riemann surface M, induces the manifold topology.

Proof It is an easy exercise to check that the only difficulty we might encounter would be the existence of an open neighbourhood U, of a zero of g, z say, which does not contain a δg-open neighbourhood of z.

So, suppose that ξ is a local coordinate centred at z : there exists $\varepsilon > 0$ such that $B_E(0,\varepsilon) = \{\xi \; ; \; |\xi| < \varepsilon\}$ lies in U and is relatively compact in M. Let $\delta = \inf_{|\xi| = \frac{\varepsilon}{2}} \delta g(0,\xi)$: since δg is a metric and $\{ \xi \; ; \; |\xi| = \frac{\varepsilon}{2} \}$ is compact, we have $\delta > 0$. We now show that the δg-ball at z of radius δ, lies in $B_E(0,\frac{\varepsilon}{2})$ and hence lies in U.

Suppose that $|\xi| \geqslant \frac{\varepsilon}{2}$. If $|\xi| = \frac{\varepsilon}{2}$ then $\delta g(0,\xi) \geqslant \delta$, by definition. If $|\xi| > \frac{\varepsilon}{2}$ then, since $\{ \xi \; ; \; |\xi| = \frac{\varepsilon}{2} \}$ is complete, any continuous path from 0 to ξ must intersect it at some point,

ξ' say. So, if γ is a piece-wise smooth path from 0 to ξ, we have

$$\int_\gamma ds_g \geqslant \delta_g(0,\xi') \geqslant \delta$$

and hence $\delta_g(0,\xi) \geqslant \delta$. So, if $\delta_g(0,\xi) < \delta$ then $|\xi| < \frac{\varepsilon}{2}$.

The proof of the following proposition is left as an exercise for the reader :

Proposition 1.6A Suppose that g is a (1,1) semi-metric on a Riemann surface M whose set of zeros Z, is finite. If every divergent path on M has infinite length then $(M,\delta g)$is a complete metric space.

(B) Null Holomorphic Curves in C^3.

A map $\phi : M \longrightarrow \mathbb{R}^3$ is harmonic iff $\frac{d\phi}{d\xi}\, d\xi$ is a holomorphic 1-form on M and hence if M is simply-connected then

$$\Omega(\xi) = \int^{\xi} \frac{d\phi}{d\xi}\, d\xi$$

describes a holomorphic curve in C^3 such that $\phi = 2\mathrm{Re}(\Omega)$. (We are following the convention here, that by this equation we mean that $\phi(\xi) = \phi(\xi_0) + \int_{\xi_0}^{\xi} \frac{d\phi}{d\zeta} d\xi$, for any choice of base point $\xi_0 \varepsilon M$.) Let

$< , >^C$ denote the complex bilinear extension to C^3 of $< , >$ and observe that since $\frac{d\phi}{d\xi} = \frac{d\Omega}{d\xi}$ we have

$$4 \left\langle \frac{d\Omega}{d\xi}, \frac{d\Omega}{d\xi} \right\rangle^C = \left|\frac{d\phi}{dx}\right|^2 - \left|\frac{d\phi}{dy}\right|^2 - 2i \left\langle \frac{d\phi}{dx} , \frac{d\phi}{dy} \right\rangle$$

where $\xi = x + iy$. Consequently, $<\Omega',\Omega'>^C = 0$ iff ϕ is weakly conformal (where ' denotes differentiation with respect to ξ).

Remark 1.7 Observe that $ds_\phi^2 = Re2\Omega^* < , \bar{\ } >^C = Re2\left|\frac{d\Omega}{d\xi}\right|^2 d\xi \otimes d\bar{\xi}$, i.e. ds_ϕ^2 is the Riemannian semi-metric associated to the Hermitian semi-metric induced on M by Ω.

Definition 1.8 (i) A vector $\nu \varepsilon C^3$ such that $<\nu,\nu>^C = 0$, is said to be null. *The collection of such vectors forms the* null cone K, in C^3 , and an affine line with null direction is called a null line.

(ii) A holomorphic curve $\Omega : M \longrightarrow C^3$, such that $<\Omega',\Omega'>^C = 0$ is said to be null.

Theorem 1.9 If M is a simply-connected Riemann surface, then $\phi : M \longrightarrow \mathbb{R}^3$ is a branched minimal immersion iff ϕ is the real part of a non-constant, null holomorphic curve Ω, in C^3.

Now, $K\setminus\{0\}$ is simply the pre-image of the quadric, Q_1, under the projection of $C^3\setminus\{0\}$ to complex projective space, $\mathbb{P}_2$. Hence if $\Omega : M \longrightarrow C^3$, is a non-constant, null holomorphic curve, then there is associated to Ω the holomorphic map $\gamma_\Omega : M\setminus B_\Omega \longrightarrow Q_1$ given by $\gamma_\Omega(\xi) = [\Omega'(\xi)]$, where $B_\Omega = \{\xi \in M; \Omega'(\xi) = 0\}$ is the branch set of Ω. Furthermore, γ_Ω extends analytically over B_Ω : for, suppose that $\xi_0 \in B_\Omega$, since Ω is holomorphic there exists $n \in \mathbb{N}$ such that in a local holomorphic coordinate ξ around ξ_0,

$$\Omega'(\xi) = (\xi-\xi_0)^n\Omega_0(\xi) \qquad \text{where} \quad \Omega_0(\xi_0) \neq 0 \quad ;$$

consequently we may define $\gamma_\Omega(\xi_0) = [\Omega_0(\xi_0)]$. In the next section we provide a geometric interpretation of γ_Ω.

(C) The Gauss Map and the Weierstrass Representation Formulae.

A unit vector $u \in \mathbb{R}^3$, may be completed to a positively oriented, orthonormal basis, $\{u_1,u_2,u\}$ say, and if S^2, the unit sphere at the origin of $\mathbb{R}^3$, is endowed with the orientation induced by the inward pointing unit normal field, then it is easy to check that $D : S^2 \longrightarrow Q_1$ given by $D(u) = [u_1 - iu_2]$

is a conformal diffeomorphism. If $s : S^2\setminus\{0,0,1\} \longrightarrow C$, denotes stereographic projection to the $(x,y,0)$-plane, then there is the chart $\chi = D \circ s^{-1}$, on $Q_1\setminus\{[1,-i,0]\}$ given by $\chi(\zeta) = [1-\zeta^2, i(1+\zeta^2), 2\zeta]$. In fact, χ may be viewed as a restriction of the biholomorphism

$$q : \mathbb{P}_1 \longrightarrow Q_1 \quad \text{given by} \quad q([\zeta_0,\zeta_1]) = [A(\zeta_0,\zeta_1)]$$

where

$$A(\zeta_0,\zeta_1) = -\tfrac{1}{2}(\zeta_1^2 - \zeta_0^2, i(\zeta_0^2+\zeta_1^2), 2\zeta_0\zeta_1) .$$

<u>Definition 1.10</u> For a smooth branched immersion $\phi : M \longrightarrow \mathbb{R}^3$, let $\gamma_\phi : M\setminus B_\phi \longrightarrow S^2$ denote the Gauss map of ϕ, and let $g = s \circ \gamma_\phi$.

If $\phi : M \longrightarrow \mathbb{R}^3$ is non-constant and weakly conformal then on $M\setminus B_\phi$ we have

$$D \circ \gamma_\phi(\xi) = \left[\frac{d\phi}{dx}(\xi) - i\frac{d\phi}{dy}(\xi)\right] = \left[\frac{d\phi}{d\xi}(\xi)\right] .$$

Hence ϕ is a branched minimal immersion iff $\gamma_\phi = D^{-1} \circ \left[\frac{d\phi}{d\xi}\right]$ is holomorphic. In particular, observe that for such a ϕ, γ_ϕ extends over B_ϕ.

<u>Remark 1.11</u> If $\Omega : M \longrightarrow C^3$ is a non-constant, null holomorphic curve, then γ_Ω may be interpreted as the Gauss map of $\mathrm{Re}(\Omega)$. Consequently we will refer to γ_Ω as the <u>Gauss map of Ω</u>.

If $\phi : M \longrightarrow \mathbb{R}^3$ is a branched minimal immersion then $\chi \circ g = D\circ\gamma_\phi$, hence there exists a holomorphic function F on M, such that

$$\frac{d\phi}{d\xi} = \frac{F}{2}(1-g^2, i(1+g^2), 2g) \quad .$$

Observe that the zeros of F correspond with the branch points of ϕ. Integrating these equations gives the following Weierstrass representation formulae :

$$\phi_1(\xi) = \mathrm{Re}\{\tfrac{1}{2}\int^{\xi} F(1-g^2)d\xi\}$$

$$\phi_2(\xi) = \mathrm{Re}\{\frac{i}{2}\int^{\xi} F(1+g^2)d\xi\}$$

$$\phi_3(\xi) = \mathrm{Re}\{\int^{\xi} Fg\, d\xi\} \quad .$$

Clearly, if $\phi = \mathrm{Re}(\Omega)$ then in order to write down the corresponding formulae for Ω, one simply deletes 'Re'.

Suppose that $\Omega : M \longrightarrow C^3$ is null and $g'(\xi_0) \neq 0$. There exists an inverse, g^{-1}, on a neighbourhood U, of $\zeta_0 = g(\xi_0)$; $\xi = g^{-1}(\zeta)$ gives $d\xi = \frac{dg^{-1}}{d\zeta}d\zeta$, so if f is a holomorphic function on U which satisfies

$$f'''(\zeta) = F\circ g^{-1}(\zeta)\frac{dg^{-1}}{d\zeta}(\zeta) \quad ,$$

then

$$\Omega_1 \circ g^{-1}(\zeta) = \tfrac{1}{2}\int^{\zeta} f'''(\zeta)(1-\zeta^2)d\zeta$$

$$\Omega_2 \circ g^{-1}(\zeta) = \frac{i}{2}\int^{\zeta} f'''(\zeta)(1+\zeta^2)d\zeta$$

$$\Omega_3 \circ g^{-1}(\zeta) = \int^{\zeta} f'''(\zeta)\zeta\, d\zeta \quad , \text{ for } \zeta \in U .$$

Integrating these equations by parts yields the following Weierstrass representation formulae in free form for Ω on U :

$$\Omega_1 \circ g^{-1}(\zeta) = \frac{1}{2}(1-\zeta^2)f''(\zeta) + \zeta f'(\zeta) - f(\zeta)$$

$$\Omega_2 \circ g^{-1}(\zeta) = \frac{i}{2}(1+\zeta^2)f''(\zeta) - i\zeta f'(\zeta) + if(\zeta)$$

$$\Omega_3 \circ g^{-1}(\zeta) = \zeta f''(\zeta) - f'(\zeta) \quad .$$

These formulae were first announced in 1866 by Weierstrass, see [W] and [E] : in §3 we provide a geometric interpretation of f, which is derived from [H2]; for some simple examples see §1.E.

Remark 1.12 (i) The substitution of any holomorphic function into the above formulae yields a null holomorphic curve in C^3 ; in particular, note that quadratic functions give constant maps.

(ii) The description of a null holomorphic curve Ω, via these formulae is very special, in that the curve is parameterized by its Gauss map, i.e. $g(\zeta) = \zeta$. A consequence of this is that the collection of null curves on $U \subset C$, described by the above formulae, possesses a complex vector space structure, There is nothing mysterious about this : if $\Omega, \Psi : U \longrightarrow C^3$ are null, then

$$<\Omega' + \Psi', \Omega' + \Psi'>^C = 2<\Omega', \Psi'>^C \quad ,$$

and this is the only obstruction to the nullity of $\Omega + \Psi$. But, if $g_\Omega(\zeta) = g_\Psi(\zeta) = \zeta$, then $\Omega'(\zeta)$ and $\Psi'(\zeta)$ lie on the same null line in C^3 and hence $<\Omega'(\zeta), \Psi'(\zeta)>^C = 0$.

(iii) Given a branched minimal immersion $\phi : M \longrightarrow \mathbb{R}^3$, it would be natural to determine precise geometric conditions which ensure that ϕ has a Weierstrass representation in free form in some neighbourhood of a point $\xi_0 \varepsilon M$. Clearly $g'(\xi_0) \neq 0$ is a sufficient condition, but in fact is not necessary ; see §3.D for further details.

(iv) Observe that $f'''(\zeta) = 0$ iff ζ is a branch point of Ω.

(v) Suppose that f generates Ω_f via the Weierstrass formulae ; for $\alpha \varepsilon C$, observe that $\Omega_{\alpha f} = \alpha\Omega_f$. Consequently, if $\phi = \mathrm{Re}(\Omega)$ then the associate surfaces $\phi_\theta = \mathrm{Re}(e^{i\theta}\Omega_f)$, where $0 < \theta < \pi$, correspond with $e^{i\theta}f$. So αf generates a null curve in C^3 which is simply a scaling of an associate curve of Ω_f.

Furthermore, if $\tilde{f}(\zeta) = f(\zeta) + a\zeta^2 + b\zeta + c$, for $a,b,c \varepsilon C$, then

$$\Omega_{\tilde{f}} = \Omega_f + (a-c,i(a+c),-b) \quad .$$

So the addition of a quadratic function to f corresponds to translating Ω_f in C^3. A full understanding, in terms of f, of the action of the Euclidean group on a minimal surface, may be obtained from the interpretation in §3.

(D) Gaussian Curvature.

Recall that if h is an Hermitian metric on M, then $\Phi_h = -\frac{1}{2}\mathrm{Im}(h)$, is the associated (1,1)-form, and furthermore if Θ_h denotes the curvature 2-form of h, then the Cartan structure equation is simply

$$i\Theta_h = K_h\Phi_h ,$$

where K_h denotes the Gaussian curvature of Re(h). If, in a local coordinate ξ, $h_\xi = 2\lambda(\xi)d\xi \otimes d\bar{\xi}$ then

$$\Phi_h(\xi) = i\lambda(\xi)d\xi \wedge d\bar{\xi} \quad \text{and}$$

$$K_h(\xi) = -\frac{1}{\lambda(\xi)} \frac{d}{d\xi}\frac{d}{d\bar{\xi}} \log(\lambda(\xi)) , \qquad \text{thus}$$

$$\Theta_h(\xi) = \partial\bar{\partial}\log(\lambda(\xi)) \quad : \quad \text{see [G\&H] for details.}$$

Now, suppose that $h = \Omega^*\langle\ ,\bar{\ }\rangle^c$, where $\Omega : M \longrightarrow C^3$ is null holomorphic, and let Θ_Ω denote the curvature 2-form.

Proposition 1.13 Θ_Ω is finite on M.

Proof The only problem may occur at branch points. If ξ_0 is a branch point of Ω, then $\Omega'(\xi) = (\xi-\xi_0)^n\Omega_0(\xi)$ for some $n \in \mathbb{N}$, when $\Omega_0(\xi_0) \neq 0$. So, on $U\setminus\{\xi_0\}$

$$\Theta_\Omega(\xi) = \frac{d}{d\xi}\frac{d}{d\bar{\xi}}\log[|(\xi-\xi_0)^n\Omega_0(\xi)|^2]d\xi \wedge d\bar{\xi}$$

$$= \frac{d}{d\xi}\frac{d}{d\bar{\xi}}\log(|\Omega_0(\xi)|^2)d\xi \wedge d\bar{\xi} , \quad \text{which extends over U.}$$

It is instructive to calculate the Gaussian curvature of the metric induced by a null curve on a domain $\mathbb{P}_1$, after it has been reparameterized by its Gauss map, g, i.e. to use g as a local coordinate chart on M. So, suppose that $g'(\xi_0) \neq 0$ and g^{-1} is an inverse on a neighbourhood $U' = g(U)$ of $\zeta_0 = g(\xi_0)$. If ζ is the local coordinate on U' then since $g(\zeta) = \zeta$,

$$\frac{d\Omega}{d\zeta}(\zeta) = \frac{F(\zeta)}{2}(1-\zeta^2,\ i(1+\zeta^2),\ 2\zeta)$$

and $f'''(\zeta) = F(\zeta)$. Hence

$$\left|\frac{d\Omega}{d\zeta}(\zeta)\right|^2 = \frac{1}{2}|f'''(\zeta)|^2(1+|\zeta|^2)^2 \quad ,$$

consequently the Gaussian curvature of $\mathrm{Re}(\Omega)$ is given by

$$K_\Omega(\zeta) = -\frac{1}{\frac{1}{2}|f'''(\zeta)|^2(1+|\zeta|^2)^2}\frac{d}{d\zeta}\frac{d}{d\bar{\zeta}}\log(\tfrac{1}{2}|f'''(\zeta)|^2(1+|\zeta|^2)^2) \ .$$

The key point here is that since f''' is holomorphic, we have

$$\frac{d}{d\zeta}\frac{d}{d\bar{\zeta}}\log(\tfrac{1}{2}|f'''(\zeta)|^2(1+|\zeta|^2)^2) = 2\frac{d}{d\zeta}\frac{d}{d\bar{\zeta}}\log(1+\zeta\bar{\zeta})$$

$$= \frac{2}{(1+|\zeta|^2)^2}$$

Thus, $K_\Omega(\zeta) = -\dfrac{4}{|f'''(\zeta)|^2(1+|\zeta|^2)^4}$.

Now, let $C_\Omega(U)$ denote the total Gaussian curvature on U, i.e. $C_\Omega(U) = \iint_U i\Theta_\Omega$ and let $Ag(U)$ denote the area induced on U by the Gauss map. The following classical result is particularly easy to prove using g as a local chart :

<u>Theorem 1.14</u> $C_\Omega(M) = -Ag(M)$.

<u>Proof</u> Since B_g consists of isolated points, it has zero measure with respect to $\mathrm{Re}(\Omega^* < \,,\,\bar{}\,>^c)$, furthermore $M \backslash B_g$ is a countable union of open subsets on which g has an inverse. On such a U we have from above that, in the chart ζ given by g ,

$$i\Theta(\zeta) = -\frac{2i\, d\zeta \wedge d\bar{\zeta}}{(1+|\zeta|^2)^2} ,$$

which is simply minus the density associated to the metric of constant curvature 1 on $\mathbb{P}_1$, and hence

$$\iint_U i\Theta = -\text{Area of } g(U) .$$

Thus

$$C_\Omega(M) = -Ag(M) .$$

<u>(E) Examples</u>

In this section we give some simple examples of branched minimal surfaces in $\mathbb{R}^3$ generated by the Weierstrass formulae in free form.

<u>Example 1.15</u> (Enneper's surface) Substituting $f(\zeta) = \frac{1}{6}\zeta^3$ into the Weierstrass formulae yields :

$$\Omega_1(\zeta) = \frac{1}{2}\zeta - \frac{1}{6}\zeta^3$$

$$\Omega_2(\zeta) = \frac{i}{2}\zeta + \frac{i}{6}\zeta^3$$

$$\Omega_3(\zeta) = \frac{1}{2}\zeta^2 \quad .$$

Hence if $\phi = \mathrm{Re}(\Omega)$ then

$$\phi_1(x,y) = \frac{1}{2}\left(x - \frac{y^3}{3} + xy^2 \right)$$

$$\phi_2(x,y) = \frac{1}{2}\left(-x + \frac{y^3}{3} - x^2 y \right)$$

$$\phi_3(x,y) = \frac{1}{2}\left(x^2 - y^2 \right) \quad .$$

Thus, f generates Enneper's surface. It is easy to check that this surface is complete and has $C_\phi(C) = -4\pi$.

Proposition 1.16 A minimal surface $\phi : C \longrightarrow \mathbb{R}^3$, that is complete, free of branch points and generated by an entire function, f, is a scaled associated surface of Enneper.

Proof The following lemma is 9.6 of [Os 1] and is due to Finn [F].

Lemma Let $F(\zeta)$ be analytic and different from zero for $0 < R < |\zeta| < \infty$. Suppose that for every path ρ, which diverges to infinity we have

$$\int_\rho |F(\zeta)|\,|d\zeta| = \infty \quad .$$

Then $F(\zeta)$ has at most a pole at infinity.

Since the metric is complete, $\int_\rho |f'''(\zeta)|(1+|\zeta^2|)|d\zeta| = \infty$ for any ρ which diverges to infinity. Hence $\int_\rho |\zeta^2 f'''(\zeta)||d\zeta| = \infty$ for any such ρ, consequently $\zeta^2 f'''(\zeta)$ is an entire function with at most a pole at infinity and is therefore, a polynomial. So $f'''(\zeta) = (a_0 + a_1\zeta + \ldots + a_n\zeta^n)/\zeta^2$, and it is clear that if a_0 and a_1 are not both zero then f cannot be entire. Thus $f'''(\zeta)$ is polynomial. Since ϕ is free of branch points, $f'''(\zeta) \neq 0$ for $\zeta \in C$ which implies that $\deg(f) \leqslant 3$, i.e. $f(\zeta) = a\zeta^3 + b\zeta^2 + c\zeta + d$. From Remark 1.12(v), Ω_f is a translate of $re^{i\theta}\Omega_{\zeta^3}$, hence $\phi = \mathrm{Re}(\Omega_f)$ is simply a translate in $\mathbb{R}^3$ of $r\mathrm{Re}(e^{i\theta}\Omega_{\zeta^3})$.

Example 1.17 Lawson gives in equation 2.17 of [L] (which contains a slight printing error), the following example of a minimal surface in $\mathbb{R}^3$, which is branched at $z = 0$:

$$\begin{aligned}\psi(z) &= (\mathrm{Re}(z^2 - \tfrac{1}{2}\bar{z}^4),\ \mathrm{Im}(z^2 - \tfrac{1}{2}\bar{z}^4),\ \mathrm{Re}(\tfrac{4}{3}z^3)) \\ &= \mathrm{Re}((z^2 - \tfrac{1}{2}z^4),\ -i(z^2 + \tfrac{1}{2}z^4),\ \tfrac{4}{3}z^3)\end{aligned}$$

This is simply the reflection, in the (x,z)-plane, of the surface generated by $f(z) = \frac{1}{6}z^4$, for :

$$\begin{aligned}\Omega_1(z) &= (1-z^2)z^2 + \tfrac{2}{3}z^4 - \tfrac{1}{6}z^4 &&= z^2 - \tfrac{1}{2}z^4 \\ \Omega_2(z) &= i(1+z^2)z^2 - \tfrac{2}{3}iz^4 + \tfrac{i}{6}z^4 &&= i(z^2 + \tfrac{1}{2}z^4) \\ \Omega_3(z) &= 2z^3 - \tfrac{2}{3}z^3 &&= \tfrac{4}{3}z^3 \quad .\end{aligned}$$

The surface is complete and has $C_\phi(C) = -4\pi$. It is also easy to see that, since the Gauss map of Φ does not branch at z=0, Φ has a genuine branch point there, i.e. the branching is not a result of the parameterization, (see Appendix A).

Example 1.18 Fourth order perturbation of Enneper's surface :

If $$f(\zeta) = \frac{\zeta^3}{6} + \frac{\varepsilon\zeta^4}{12}$$

then $$\Omega_f = \Omega_{\frac{\zeta^3}{6}} + \Omega_{\frac{\varepsilon\zeta^4}{12}} \quad .$$

$\phi = \mathrm{Re}(\Omega)$ is a complete minimal surface in $\mathbb{R}^3$, with $C_\phi(C) = -4\pi$; and since $f'''(\zeta) = 1 + \varepsilon\zeta$, ϕ has a branch point at $\zeta = -\frac{1}{\varepsilon}$. Thus as $\varepsilon \longrightarrow 0$, the surfaces 'converge to Enneper', (the branch point 'runs off to infinity').

Example 1.19 Given $\zeta_1, \ldots, \zeta_n \in C$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, let

$$f'''(\zeta) = \prod_{K=1}^{n} (\zeta - \zeta_K)^{\alpha_K} \quad .$$

Since $\Omega_f'(\zeta) = (1-\zeta^2,\ i(1+\zeta^2),\ 2\zeta)f'''(\zeta)$, $\mathrm{Re}(\Omega_f)$ is a branched minimal surface with a branch point of order α_K at ζ_K, for $K = 1, \ldots, n$.

Example 1.20 In [B&C] we are given explicit formulae for Henneberg's surface via a double cover of $C\backslash\{0\}$ onto the image in $\mathbb{R}^3$, which has the topology of a Mobius strip.

The Weierstrass representation for this surface is given there by

$$g(\zeta) = \zeta \quad \text{and} \quad F(\zeta) = 2\left(1 - \frac{1}{\zeta^4}\right) \quad .$$

It is therefore very easy to solve $f'''(\zeta) = 2\left(1 - \frac{1}{\zeta^4}\right)$, to obtain

$f(\zeta) = \frac{1}{3}\left(\frac{1}{\zeta} + \zeta^3\right)$. Note that this surface has branch points at $\{\pm 1, \pm i\}$.

Example 1.21 It is easy to check that $f(\zeta) = -\zeta\log(\zeta)$ on $C\backslash\{\text{negative real axis}\}$ generates the Catenoid.

These examples are discussed again in §4 where we give some examples where the Gauss map is not bijective.

(F) Complete Branched Minimal Surfaces of Finite Total Gaussian Curvature.

In this section we briefly review some of the fundamental ideas of Ossermann, see [Os 1], [Os 2], and Jorge and Meeks [J&M]. Their work provides us with a good understanding of the global structure of complete minimal surfaces in $\mathbb{R}^3$ of finite total Gaussian curvature. We indicate why their results are essentially unaffected by the presence of a finite number of branch points. The motivation here is that the algebraic minimal surfaces (we describe in §4) fall within this class, and it is unnatural in the theory we describe in §§2,3 to restrict attention to immersed surfaces.

Definition 1.22 Suppose that M is a semi-Riemannian surface. Let $\mathfrak{R}_M$ denote the collection of regions of M such that $R \in \mathfrak{R}_M$ iff

(i) R is homeomorphic to a punctured disc.

(ii) There exists a connected subset $R' \subset R$ such that $R \setminus R'$ is an annulus and the closure of $R \setminus R'$ is compact.

(iii) Every path on R which diverges to the puncture point has infinite length.

Deeming two such regions of M to be equivalent iff their union lies in $\mathfrak{R}_M$ we obtain the collection of equivalence classes $\mathfrak{R}_M/\sim$, known as the <u>ends of M</u> .

<u>Remark 1.23</u> Observe that if M is complete and finitely connected then $|\mathfrak{R}_M/\sim|$ is the number of boundary components of M.

The Gauss map of branched minimal immersions lying in the following class of surfaces is well behaved at the ends, i.e. these surfaces have well-defined normal vectors at infinity.

<u>Definition 1.24</u> Let $\phi : M \longrightarrow \mathbb{R}^3$ be a branched minimal immersion. The Gauss map $\gamma_\phi : M \longrightarrow \mathbb{P}_1$ is called <u>algebraic</u> if

(a) M is conformally equivalent to a compact Riemann surface M', punctured at a finite number of points, i.e. M is parabolic.

(b) γ_ϕ extends to a meromorphic function $M' \longrightarrow \mathbb{P}_1$.

We now describe geometric criteria for ϕ which ensure that γ_ϕ is algebraic. First we need the following proposition which is a simple adaptation of Proposition 16 of [L : III.6].

<u>Proposition 1.25</u> If $\phi : M \longrightarrow \mathbb{R}^3$ is a complete, branched minimal immersion of finite total Gaussian curvature with $|B_\phi| < \infty$, then M is a parabolic Riemann surface.

<u>Proof</u> We have (a) $K_\phi \leqslant 0$ and (b) $\iint_M |K_\phi| dA_\phi < \infty$. Now, clearly we can construct a genuine metric g on M that agrees with ds^2_ϕ on the compliment of a compact subset M_o which contains Z. Since g is uniformly equivalent to the Euclidean metric on coordinate charts in M_o, and agrees with ds^2_ϕ on $M \setminus M_o$, g is complete. Furthermore (b) implies that $\iint_M |K_g| dA_g < \infty$ since $|K_g| dA_g = |K_\phi| dA_\phi$ on $M \setminus M_o$. It follows from a theorem of Huber, see [Hu], that M is finitely connected. Consequently, M_o is bounded by a finite number of Jordan curves $\gamma_1, \ldots, \gamma_r$ such that each component M_j, of $M \setminus M_o$ can be conformally mapped onto the annulus $D_j = \{z \in C ; 1 < |z| < r_j\}$, see

[A&S : I44D, IIB]. Since $Z \subset M_0$ the delicate potential theoretic arguements, which demonstrate that completeness together with (a) and (b) implies that each $r_j = \infty$, are unaffected, see [L : III.6]. Hence we can conformally attach the discs $D_j \cup \{\infty\}$ to M via the maps $M_j \simeq D_j$ to give a compact Riemann surface $M' \supset M$.

Having established this proposition an inspection of the proof of theorem 12 of [L : III.6] reveals that the following is a straightforward generalization :

Theorem 1.26 If $\phi : M \longrightarrow \mathbb{R}^3$ is a complete branched minimal immersion such that B_ϕ is finite then

$$C_\phi(M) > -\infty \qquad \text{iff } \gamma_\phi \text{ is algebraic.}$$

Consequently we have the usual quantization of total Gaussian curvature :

Corollary 1.27 If $\phi : M \longrightarrow \mathbb{R}^3$ is a complete branched minimal immersion with B_ϕ finite then $C_\phi(M) = -\infty$ or $C_\phi(M) = -4\pi k$, where $k = \deg(\gamma_\phi)$.

Remark 1.28 (i) It is necessary to suppose that B_ϕ is finite ; the assumption of completeness together with finite total Gaussian curvature is not sufficient, for there exist complete branched minimal immersions of the disc into $\mathbb{R}^3$ the images of whose Gauss maps have arbitarily small area, see [Os 1].

(ii) If $\phi : M \longrightarrow \mathbb{R}^3$ is complete and $|B_\phi| < \infty$ then γ_ϕ is algebraic iff $\frac{d\phi_j}{d\xi} d\xi$ are meromorphic 1-forms on M'.

We now turn to Ossermann's inequality, see [Os 2], which represents a considerable refinement of the classical Cohn - Vossen inequality. We provide a proof which is slightly different to the usual version and we allow for the possibility of (finite) branching of ϕ.

Suppose that M is a compact Riemann surface and $\phi : M\backslash\{p_1, .. , p_r\} \longrightarrow \mathbb{R}^3$ is such that $\alpha = \frac{d\phi}{d\xi} d\xi$ is a C^3-valued meromorphic 1-form on M. Let $D_\phi = D_0 + D_\infty$ denote the divisor of zeros and poles of α, and observe that for a coordinate ξ centred at $p \in D_\phi$ we have

$$\left|\frac{d\phi}{d\xi}\right|^2 = |\xi|^{2n} |\mu(\xi)|^2$$

where $n = \mathrm{ord}_p(\alpha)$ and $\mu(0) \neq 0$ or ∞. Consequently, identifying sections of $T'_M \otimes D_\phi$ with meromorphic fields θ on M such that

$$\mathrm{ord}_p(\theta) \geqslant -\mathrm{ord}_p(\alpha)$$

observe that ds^2_ϕ gives a genuine metric on $T'_M \otimes D_\phi$.

<u>Definition 1.29</u> For a complete branched minimal immersion $\phi : M\backslash\{p_1, .. , p_r\} \longrightarrow \mathbb{R}^3$ with γ_ϕ algebraic, let $\beta_\phi = \deg(D_0)$ denote the <u>total branching order of ϕ</u>.

Proposition 1.30 If $\phi : M\setminus\{p_1, .. , p_r\} \longrightarrow \mathbb{R}^3$ is a complete finitely branched minimal surface with γ_ϕ algebraic then

$$C_\phi(M\setminus\{p_1, \ldots ,p_r\}) = 2\pi(\chi(M) + \beta_\phi + \deg(D_\infty))$$

Proof Follows immediately from Chern - Weil formula :

$$\int_M \frac{i}{2\pi}\theta = \deg(T_M' \otimes D_\phi) .$$

It follows from the single-valued nature of ϕ that the order of each pole of α is greater than 1, see lemma 18 of [L : III.6]. Hence we have

Corollary 1.31 If $\phi : M\setminus\{p_1, .. , p_r\} \longrightarrow \mathbb{R}^3$ is a complete branched minimal immersion with B_ϕ finite then

$$C_\phi(M\setminus\{p_1, \ldots ,p_r\}) \leqslant 2\pi(\chi(M) + \beta - 2r) .$$

Note that r equals the number of ends and hence $\chi(M) - r$ is the Euler character of $M\setminus\{p_1, .. , p_r\}$. (If $\beta = 0$ we recover the usual Ossermann inequality.)

Remark 1.32 Observe that Ossermann's formula with branching is very 'flabby', in that we may reparameterize our image surface $\phi(M\setminus\{p_1, .. , p_r\})$, by composing with an arbitary holomorphic map $\tilde{M} \xrightarrow{\theta} M$ and hence increase all the data associated to this image. (Of course we cannot do that and keep $\beta = 0$.) We solve the problem of how to describe a 'minimal' parameter domain for a certain class of branched minimal surfaces in §4. (This type of problem has been considered in [G - O - R] but our ideas arise out of quite different considerations.)

Note that the introduction of a finite number of branch points does generate new geometric phenomena, as the following simple example shows :

Example 1.33 Consider $\phi_f = \mathrm{Re}\Omega_f : C\setminus\{\zeta_1, .. , \zeta_n\} \longrightarrow \mathbb{R}^3$ where

$$f(\zeta) = \sum_{\ell=1}^{n} (\zeta - \zeta_\ell)^{-1} .$$

Since the genus of $\mathbb{P}_1$ is 0 and $C_{\phi_f}(C\setminus\{\zeta_1, .. , \zeta_n\}) = -4\pi$, the usual Ossermann inequality would force the number of ends to be 1 or 2. Of course, $r = n$ can be made arbitarily large in this class of examples.

The following beautiful picture of the global structure of a complete minimal immersion with finite total Gaussian curvature appears in [J&M] :

Theorem 1.34 Suppose that $\phi : M \longrightarrow \mathbb{R}^3$ is a complete minimal immersion with $C(M) > -\infty$ and let $X_n = \frac{1}{n}(\phi(M) \cap S^2(n))$, where $S^2(n)$ denotes the 2-sphere of radius n in $\mathbb{R}^3$, centred at the origin. Then

(i) ϕ is proper.

(ii) For large n, $X_n = \{\gamma_1^n, .. , \gamma_r^n\}$ consists of r immersed closed curves on S^2.

(iii) γ_i^n converges smoothly (with multiplicity) to a geodesic on S^2 as $n \longrightarrow \infty$.

Thus (in the language of [J&M]), '$\phi(M)$ viewed from infinity looks like a finite number of planes passing through the origin'. (For example, the catenoid viewed from infinity looks like two oppositely orientated copies of a plane passing through the origin.)

Now, the above theorem follows from Theorem 1 of [J&M], the proof of which is concerned solely with the behaviour of ϕ on the ends, and is therefore unaffected by the presence of a finite number of branch points. Consequently observe that we have the same picture 'from infinity', for the global structure of any complete branched minimal immersion with $|B_\phi| < \infty$.

Remark 1.35 (i) It is clear that the asymptotic flatness of the ends is really a consequence of completeness and non-compactness together with the finiteness of $\iint_M |K| dA$, i.e. the minimality is not the essential feature here.

(ii) Observe that a central feature of the global structure of a complete minimal surface of finite total curvature which emerges from the above considerations is that the topology and curvature are 'highly concentrated'. In §4 we discuss the possibility of studying families of such surfaces which exhibit soliton behaviour.

§2 The Weierstrass - Hitchin Correspondence.

(A) Introduction.

Let V be a 3-dimensional complex vector space and suppose that K is a conformal structure on V , i.e. K is a conformal equivalence class of non-degenerate symmetric bilinear forms. Recall that K is described equivalently, by the following geometric object in V, which is called the null cone, and also denoted K :

$$K = \{v \in V \ ; \ (v,v) = 0\} \text{ , where } (\ , \ ) \in K.$$

Let $\mathbb{P}(V)$ denote the projective space of lines in V, and let Q(K) denote the curve in $\mathbb{P}(V)$ of null lines, (i.e. lines that lie on K). Q(K) has genus 0, and is therefore biholomorphic to $\mathbb{P}$.

Generically, a plane π in V intersects K in a pair of null lines : this follows simply from the fact that the quadratic form associated to any $(\ , \ ) \in K$ splits on restriction to π, into a product of a pair of linear factors. If that restriction is degenerate, then π intersects K in a single line (with multiplicity 2).

Our purpose in §2.B is to describe the geometry of the collection, Π, of affine translates of planes in V that intersect K in a single line. We show that Π has the structure of a holomorphic line bundle on Q(K) of degree 2 , and furthermore that there exists a natural geometric correspondence between V and the global holomorphic sections of Π over Q(K).

The remaining sections of this chapter consist of results which enable us, via the correspondence outlined in §2.B, to pass between null holomorphic curves in V, (i.e. those curves whose derivative take only null directions), and (free) holomorphic curves in Π, culminating in the correspondence stated in Theorem 2.26. The formulation given here is invariant, and consequently fairly abstract ; the hope is that this approach makes the key features of the construction clear. We discuss the introduction of coordinates in §3, where we apply the ideas of this chapter to study minimal surfaces in $\mathbb{R}^3$, and in particular give a geometric interpretation of the Weierstrass representation formulae.

Finally, observe that the most significant feature of this correspondence is perhaps, that the nullity condition, which acts as a constraint on the space of holomorphic curves in V, is removed by a 'change of background space'; it is in effect, encoded into the holomorphic geometry of Π. Hence we gain, by having transformed the data of a null curve in V into the Cauchy - Riemann equations on Π, at the cost of course, of working in a slightly less trivial ambient space. It becomes clear that the null curve we see in V is merely a manifestation of another holomorphic curve whose home is Π. This is of course, very much in the spirit of the Penrose twistor programme, see [P]. We will exploit this encoding in §4, when we come to calculate the moduli spaces of certain classes of null meromorphic curves in C^3, and branched minimal surfaces in $\mathbb{R}^3$.

(B) The Geometry of Affine Null Planes.

Definition 2.1 An affine plane π in V is said to be null, if the translate of π that passes through the origin, intersects K in a single line. If that intersection is the null line $\lambda \varepsilon Q(K)$, then π is an affine λ-plane.

In the next proposition we characterize affine λ-planes ; first we introduce the following notation : for $v \varepsilon V$ and $\lambda \varepsilon \mathbb{P}(V)$ let

$$v_0 = \{w \varepsilon V \; ; \; (w,v) = 0\}$$

where $(\; , \;) \varepsilon K$, and

$$\lambda_0 = v_0 \qquad \text{for } v \varepsilon \lambda \; .$$

Proposition 2.2 For $\lambda \varepsilon Q(K)$, λ_0 intersects K in the line λ, and is the unique plane to do so.

Proof First we show that for $v,w \varepsilon V$, if $v_0 = w_0$ then $[v] = [w]$ in $\mathbb{P}(V)$. Fix $(\; , \;) \varepsilon K$, and for $v \varepsilon V$ let $1_v \varepsilon V$ denote the dual of v under $(\; , \;)$. Since $\mathrm{Ker}(1_v) = v_0$, which by assumption equals $w_0 = \mathrm{Ker}(l_w)$, and $\dim(V/v_0) = 1$, we have $l_v = \alpha l_w$ for some $\alpha \varepsilon C^*$. Since (,) is non-degenerate this implies $v = \alpha w$ and hence $[v] = [w]$.

Now, for $\lambda, \mu \varepsilon Q(K)$ suppose that μ lies on λ_0. Either $\mu = \lambda$ or since $\dim(\lambda_0) = 2$, $\lambda_0 = \mathrm{span}\{\lambda, \mu\}$: now, $\mu \subset \lambda_0$ iff $\lambda \subset \mu_0$, and since $\mu \varepsilon Q(K)$ this implies that $\mu_0 = \mathrm{span}\{\lambda, \mu\} = \lambda_0$; hence $\lambda = \mu$.

We now show uniqueness. Suppose that π is a 2-dimensional subspace of V, whose intersection with K is the null line λ. A choice of basis $\nu, \mu \varepsilon \pi$, such that $\nu \varepsilon \lambda$, furnishes the coordinates (z_1, z_2) on π, where $(z_1, z_2) = z = z_1\nu + z_2\mu$. Accordingly

$$(z,z) = 2z_1z_2(\mu,\nu) + z_2^2(\mu,\mu) \qquad ;$$

and hence the restriction of (,) ε K to π is degenerate iff $(\mu,\nu) = 0$, i.e. $\mu \varepsilon \lambda_0$ which implies $\pi = \lambda_0$.

It follows immediately from 2.2 that for $v \varepsilon V$ and $\lambda \varepsilon Q(K)$, $v + \lambda_0$ is the unique affine λ-plane in V that passes through v. Furthermore observe that the collection of affine λ-planes in V may be identified with V/λ_0 .

Definition 2.3 Let $L \longrightarrow \mathbb{P}(V)$ be the universal bundle, i.e.

$$L = \{(p,v) \varepsilon \mathbb{P}(V) X V \ ; \ v \varepsilon p\} \ ,$$

and let Λ denote the restriction of L to Q(K).

Let $\underline{V} = Q(K) \times V$ and let

$$\Lambda_0 = \{(\lambda, v) \in \underline{V} \; ; \; v \in \lambda_0\} \; .$$

<u>Proposition 2.4</u> Λ_0 is a holomorphic subbundle of $\underline{V}$.

<u>Proof</u> It is sufficient to show that $\bar{\partial}\Gamma(\Lambda_0) \subset \Gamma(\Lambda^{0,1} \otimes \Lambda_0)$, where $\bar{\partial}$ denotes the $\bar{\partial}$-operator of the holomorphic structure of $\underline{V}$, and $\Lambda^{0,1}$ denotes the bundle of (0,1)-forms on Q(K). This follows because $\bar{\partial}$ then restricts to a $\bar{\partial}$-operator on Λ_0, and as a consequence of the Newlander - Nirenberg theorem, see [A-H-S] for example, Λ_0 possesses a holomorphic structure for which this restriction is the natural $\bar{\partial}$-operator ; hence Λ_0 is a holomorphic subbundle of $\underline{V}$.

Suppose that ζ is a local coordinate on $U \subset Q(K)$, and that $\tau(\zeta) = (\zeta, t(\zeta))$, is a holomorphic trivialization of Λ over U. If $\sigma(\zeta) = (\zeta, s(\zeta))$ lies in $\Gamma_U(\Lambda_0)$, then $(s(\zeta), t(\zeta)) = 0$ for all $\zeta \in U$, where $(\; , \;) \in K$; thus $\frac{d}{d\bar{\zeta}}(s,t) = (\frac{ds}{d\bar{\zeta}}, t) = 0$ on U, and since t is non-vanishing this implies that $\frac{ds}{d\bar{\zeta}}(\zeta) \in (\Lambda_0)_\zeta$, for all $\zeta \in U$. Consequently, $\bar{\partial}\sigma \in \Gamma_U(\Lambda^{0,1} \otimes \Lambda_0)$. Finally, for $\sigma \in \Gamma(\Lambda_0)$ observe that $\bar{\partial}\sigma(p) \in (\Lambda^{0,1} \otimes \Lambda_0)_p$ for all $p \in Q(K)$ follows by restricting to a coordinate neighbourhood of p.

<u>Remark 2.5</u> The line bundle $\underline{V}/\Lambda_0 \longrightarrow Q(K)$ may be viewed as the totality of affine null planes in V, where $(\underline{V}/\Lambda_0)_\lambda$ is identified with the collection of affine λ-planes. $\underline{V}/\Lambda_0$ possesses a unique holomorphic structure and from 2.4 this arises as the quotient structure.

Now, Liouville's theorem implies that the global holomorphic sections of $\underline{V}$ are of the form $S_v(\lambda) = (\lambda,v)$, for some fixed $v \in V$: the section S_v may be thought of geometrically as consisting of the quadric of affine null directions at v. The projection map $p : \underline{V} \longrightarrow \underline{V}/\Lambda_0$ induces, by composition, the map

$$H^0(Q(K),\mathcal{O}(\underline{V})) \longrightarrow H^0(Q(K),\mathcal{O}(\underline{V}/\Lambda_0))$$
$$S_v \longrightarrow \sigma_v \quad ; \quad \sigma_v(\lambda) = v + \lambda_0 \quad :$$

geometrically σ_v may be thought of as the quadric of affine null planes in V which pass through v;(p may be viewed as the analogue on $\underline{V}$, of the geodesic flow fibring the unit tangent sphere bundle of $\mathbb{R}^3$ over the space of oriented lines.). In consequence then, the map

$$\tau : V \longrightarrow H^0(Q(K),\mathcal{O}(\underline{V}/\Lambda_0))$$
$$v \longrightarrow \sigma_v$$

may be thought of as replacing $v \in V$ by the Q(K) of affine null planes which pass through it : we now show that every global holomorphic section of $\underline{V}/\Lambda_0$ arises in this way, i.e.

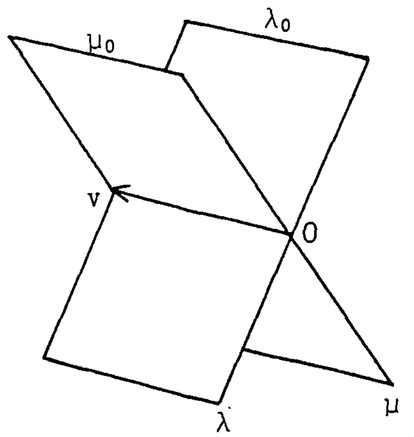
Theorem 2.6 τ is a canonical isomorphism.

Proof Since τ is obviously linear and injective it is sufficient to show that $\dim H^0(Q(K),\mathcal{O}(\underline{V}/\Lambda_0)) = 3$. A choice of $(\ ,\ ) \in K$ furnishes an isomorphism of $\underline{V}/\Lambda_0$ with Λ^{-1}, the restriction of the dual of the hyperplane bundle on $\mathbb{P}(V)$, L^{-1}. But, recall that

$c_1(L^{-1})$ is Poincare dual to the cycle on $\mathbb{P}(V)$ carried by a hyperplane divisor and hence evaluating $c_1(\Lambda^{-1})$ on $Q(K)$ calculates the generic intersection of a hyperplane in $\mathbb{P}(V)$ with $Q(K)$. We observed in the introductory remarks that this is equal to 2 and hence $\deg(\Lambda^{-1}) = 2$. The result follows from the Riemann - Roch formulae :

$$\dim H^0(Q(K),\mathcal{O}(\Lambda^{-1})) = \deg(\Lambda^{-1}) + 1 .$$

For $v \in V\setminus\{0\}$, v_0 is a 2-dimensional subspace and hence intersects K in a pair of null lines (when counted with multiplicity), λ and μ say. Since $\lambda \subset v_0$ iff $v \in \lambda_0$, v lies on both λ_0 and μ_0 ; of course, v lies on a unique null plane which passes through the origin iff v lies on K : the generic situation is illustrated in the following diagram.



In general, given 2 distinct points $v,w \in V$, generically there exist 2 affine null planes in V which pass through both of them, and there exists a unique such plane iff $v-w \in K$. This 'coupling' which occurs in the geometry of affine null planes, and which is not present in the geometry of affine null directions, emerges in the passage from $\underline{V}$ to $\underline{V}/\Lambda_0$, manifesting itself there in the non-triviality of $\underline{V}/\Lambda_0$ as a line bundle over $Q(K)$.

It is now straightforward to describe the image of K under τ : since v lies on K iff σ_v intersects the zero section at one point, $\tau(K)$ consists of those global holomorphic sections of $\underline{V}/\Lambda_0$ which possess a double root ; in particular $\lambda \varepsilon Q(K)$ corresponds under τ, to the line in $H^0(Q(K),\mathcal{O}(\underline{V}/\Lambda_0))$, of sections which possess a double root at λ.

<u>Remark 2.7</u> (i) In 2.5 we observed that a point in $\underline{V}/\Lambda_0$ represents an affine null plane in V ; we can view this as follows : for $w \varepsilon V$ and $\mu \varepsilon Q(K)$, $\{\sigma_v \, ; \, \sigma_v(\mu) = w + \mu_0\}$ is an affine plane in $H^0(Q(K),\mathcal{O}(\underline{V}/\lambda_0))$, and it corresponds under τ to the affine null plane $w + \mu_0$ in V, for $\sigma_v(\mu) = w + \mu_0$ iff $v \varepsilon w + \mu_0$.

(ii) τ is a simple example of a Penrose twistor correspondence. It ought to be viewed in the more general context of the construction of 3-dimensional Einstein - Weyl geometries : recall that these arise as moduli spaces of holomorphic deformations of a rational curve on a complex surface, with self-intersection number 2. See [H1]for further details.

(C) The Spectral Transform of a Null Curve.

Let M be a Riemann surface and suppose that $\Omega : M \longrightarrow V$ is a non-constant, null holomorphic curve, i.e. $\partial\Omega(T'_\xi M) \in Q(K)$ for all $\xi \in M$. The map $\gamma_\Omega : M \longrightarrow Q(K)$ given by $\gamma_\Omega(\xi) = \partial\Omega(T'_\xi M)$ is the Gauss map of Ω, and is defined at the branch points of Ω in the usual way.

Definition 2.8 The spectral transform of a null curve Ω, is the map $sp(\Omega) : M \longrightarrow \underline{V}/\Lambda_0$ given by :

$$sp(\Omega)(\xi) = \Omega(\xi) + \gamma_\Omega(\xi)_0 \quad .$$

$sp(\Omega)(M) \subset \underline{V}/\Lambda_0$ is the spectral curve of Ω.

At those points $\xi \in M$ where $\partial\Omega_\xi \neq 0$, $sp(\Omega)(\xi)$ is simply the affine $\gamma_\Omega(\xi)$-plane that passes through $\Omega(\xi)$; since γ_Ω extends over the branch points of Ω, observe that $sp(\Omega)$ extends to M. $sp(\Omega)$ is a lift into $\underline{V}/\Lambda_0$ of the Gauss map, i.e. $\pi \circ sp(\Omega) = \gamma_\Omega$. Hence, note that if γ_Ω is constant, which means that the image of Ω lies on an affine null line, then the image of $sp(\Omega)$ lies in a fibre of $\underline{V}/\Lambda_0$.

Now, $sp(\Omega)$ is simply the projection, under p, of the holomorphic curve $(\gamma_\Omega, \Omega) : M \longrightarrow \underline{V}$, and hence the following is an immediate consequence of 2.4 :

Theorem 2.9 The spectral transform of a null curve in V, is a holomorphic curve in $\underline{V}/\Lambda_0$.

In the next result we see that generically, $sp(\Omega)(\xi)$ is the 2nd order tangent space to Ω at ξ.

Proposition 2.10 Let $\Omega^{(n)}(\xi)$ denote $\frac{d^n\Omega}{d\xi^n}(\xi)$ and suppose that

$\Omega^{(1)}(\xi_0) \neq 0$, $\Omega^{(2)}(\xi_0) = \ldots = \Omega^{(k-1)}(\xi_0) = 0$ and $\Omega^{(k)}(\xi_0) \neq 0$:

then

$$sp(\Omega)(\xi_0) = \Omega(\xi_0) + \mathrm{span}\{\Omega^{(1)}(\xi_0),\ \Omega^{(k)}(\xi_0)\} \quad .$$

Proof Since $(\Omega^{(1)},\Omega^{(1)})(\xi) = 0$ on M, we have

$$\frac{d^{k-1}}{d\xi^{k-1}}(\Omega^{(1)},\Omega^{(1)})(\xi) = 0 \ ,$$

hence

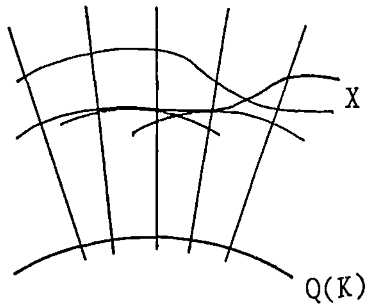
$$(\Omega^{(1)},\Omega^{(k)})(\xi) + \theta(\xi) = 0 \ ,$$

where $\theta(\xi)$ is an expression in which each term involves some $\Omega^{(n)}(\xi)$ with $2 \leqslant n \leqslant k-1$, thus $\theta(\xi_0) = 0$ and $\Omega^{(k)}(\xi_0) \in \gamma_\Omega(\xi_0)_0$.

Remark 2.11 The term 'spectral curve' is taken from [H2], where it was introduced by Hitchin to denote an algebraic curve in the holomorphic tangent bundle of $\mathbb{P}$, that is associated to a solution of the Bogomolny equations satisfying boundary conditions. We discuss the relationship between null curves and monopoles in §4.

(D) Osculating Curves in $\underline{V}/\Lambda_0$.

In this section we describe the process of 'osculating' a holomorphic curve X which lies in a holomorphic line bundle E over Q(K). The essential feature of this transformation is, very loosely, that a deformation of the zero section, and hence a curve in $H^0(Q(K),\mathcal{O}(E))$, is determined as the zero section rolls (conformally) over X. The following illustrates the sort of picture one should bear in mind here :



If $E = \underline{V}/\Lambda_0$ then the curve in $H^0(Q(K),\mathcal{O}(E))$ may be transformed via τ, to a curve in V; we show that curves obtained in this way are null.

We begin by outlining the above for the simplest case , i.e. when X is the graph of a local section of E over Q(K).

Let $\mathcal{O}$ denote the sheaf on Q(K) of germs of local holomorphic functions, and for $\lambda \varepsilon Q(K)$, let I_λ denote the ideal sheaf contained in $\mathcal{O}$, of germs of local holomorphic functions which vanish at λ. Recall that $\mathcal{O}/I_\lambda^{k+1}$ is supported at λ, and that its stalk there consists of Taylor expansions up to order k, of functions at λ.

Accordingly, if $E \longrightarrow Q(K)$ is a holomorphic line bundle then $I_\lambda^{k+1} \otimes \mathcal{O}(E)$ is the sheaf of germs of local holomorphic sections of E which vanish to order k at λ, and the quotient sheaf $\mathcal{O}(E)/I_\lambda^{k+1} \otimes \mathcal{O}(E)$ is supported at λ, the stalk there being the space of k-jets of holomorphic sections of E at λ.

Now, suppose that ζ is a holomorphic chart defined on $Q(K)\setminus\{\mu\}$ for some $\mu \in Q(K)$, (such charts exist as $Q(K) \simeq \mathbb{P}$). Since a global holomorphic section of E has precisely $d = \deg(E)$ zeros, a choice of trivialization of E over $Q(K)\setminus\{\mu\}$ enables us to identify $H^0(Q(K),\mathcal{O}(E))$ with the set of polynomials in ζ of degree $\leqslant d$. Consequently, for each $k \leqslant d$ and $\lambda \in Q(K)$ we have

$$H^0(Q(K),\mathcal{O}(E)) \xrightarrow{\eta_\lambda} \mathcal{O}(E)/I_\lambda^{k+1} \otimes \mathcal{O}(E) \longrightarrow 0 \ ,$$

and in particular, an isomorphism for $k = d$.

<u>Remark 2.12</u> The point here is that a d-jet at $\lambda \in Q(K)$, is sufficient information to determine uniquely, a global holomorphic section of E.

For any open subset of Q(K) we have, for each $\lambda \in U$, the sequence :

$$\begin{array}{ccccc} \mathcal{O}_U(E) & \longrightarrow & \mathcal{O}(E)/I_\lambda^{d+1} \otimes \mathcal{O}(E) & \xrightarrow{\eta_\lambda^{-1}} & H^0(Q(K),\mathcal{O}(E)) \\ \sigma & \longrightarrow & [\sigma]_\lambda & \longrightarrow & \eta_\lambda^{-1}([\sigma]_\lambda) \end{array}$$

Consequently, varying over $\lambda \in U$ gives a curve $\tilde{\sigma} : U \longrightarrow H^0(Q(K),\mathcal{O}(E))$, where $\tilde{\sigma}_\lambda = \eta_\lambda^{-1}([\sigma]_\lambda)$.

Proposition 2.13 For $\sigma \in \mathcal{O}_U(E)$, $\tilde{\sigma} : U \longrightarrow H^0(Q(K),\mathcal{O}(E))$ is a holomorphic curve.

Proof Without loss of generality suppose that $U \subset Q(K)\setminus\{\mu\}$, and that a trivialization of E over $Q(K)\setminus\{\mu\}$ gives the coordinates (ζ,η). Then $\sigma \in \mathcal{O}_U(E)$ is represented by the curve $\eta = f(\zeta)$, where $f \in \mathcal{O}_U$; the d-jet of σ at ζ_0 is therefore, given in these coordinates by

$$\tilde{\sigma}_{\zeta_0}(\zeta) = f(\zeta_0) + f'(\zeta_0)(\zeta-\zeta_0) + \dots + \frac{f^d(\zeta_0)}{d!}(\zeta-\zeta_0)^d .$$

Consequently the coefficients of $\tilde{\sigma}_{\zeta_0}$ are given by :

$$\begin{aligned} a_0(\zeta_0) &= f(\zeta_0) - \zeta_0 f'(\zeta_0) + \dots + (-\zeta_0)^d \frac{f^d(\zeta_0)}{d!} \\ &\ \ \vdots \\ a_d(\zeta_0) &= \frac{f^d(\zeta_0)}{d!} , \end{aligned}$$

which are holomorphic functions of ζ_0 and hence $\tilde{\sigma}$ is a holomorphic curve.

Remark 2.14 If σ is the restriction to U, of a global holomorphic section of E, then $\eta_\lambda^{-1}([\sigma]_\lambda) = \sigma$, $\lambda \in U$, and hence $\tilde{\sigma}_\lambda = \sigma$ is constant.

We now restrict attention to $E = \underline{V}/\Lambda_0$, in which case we may transform a local section $\sigma \in \mathcal{O}_U(\underline{V}/\Lambda_0)$ into the holomorphic curve $\tau^{-1} \circ \tilde{\sigma}$, in V.

<u>Theorem 2.15</u> For $\sigma \in \mathcal{O}_U(\underline{V}/\Lambda_0)$, the holomorphic curve $\tau^{-1} \circ \tilde{\sigma} : U \longrightarrow V$ is null. Furthermore, if σ is not the restriction of a global holomorphic section to U, then the Gauss map $\gamma_{\tau^{-1} \circ \tilde{\sigma}}$, is the identity map on U.

<u>Proof</u> For $\sigma \in \mathcal{O}_U(\underline{V}/\Lambda_0)$, $\tilde{\sigma}_{\zeta_0}$ is the global holomorphic section of $\underline{V}/\Lambda_0$ that satisfies the equation

$$\sigma(\zeta) = \tilde{\sigma}_{\zeta_0}(\zeta) + O[(\zeta-\zeta_0)^3] .$$

Equivalently,

$$\sigma - \tilde{\sigma}_{\zeta_0} = O[(\zeta-\zeta_0)^3]$$

and hence

$$\frac{d}{d\zeta_0} (\sigma-\tilde{\sigma}_{\zeta_0}) = O[(\zeta-\zeta_0)^2] .$$

Now, as a curve in $H^0(Q(K),\mathcal{O}(\underline{V}/\Lambda_0))$, σ is of course constant, thus we have

$$\frac{d\tilde{\sigma}_{\zeta_0}}{d\zeta_0}(\zeta) = O[(\zeta-\zeta_0)^2] .$$

Since $\tilde{\sigma}_{\zeta_0}$ is not constant, $\frac{d\tilde{\sigma}_{\zeta_0}}{d\zeta_0}$ is a global holomorphic section of $\underline{V}/\Lambda_0$ which possesses a double root at $\zeta = \zeta_0$. But, recall that $\tau(K)$ consists of those global sections which possess a double root, hence $\tau^{-1} \circ \frac{d}{d\zeta_0}\tilde{\sigma}_{\zeta_0} \in K$ and consequently $\tau^{-1} \circ \tilde{\sigma}$ is null.

Finally, observe that since $\frac{d\tilde{\sigma}_{\zeta_0}}{d\zeta_0}$ has a double root at ζ_0, it corresponds under τ to ζ_0, thus

$$\gamma_{\tau^{-1}\circ\tilde{\sigma}}(\zeta_0) = \left[\frac{d}{d\zeta_0}(\tau^{-1}\circ\tilde{\sigma}_{\zeta_0})\right] = \zeta_0 \ .$$

Our next object is to extend the above construction to any non-vertical curve in $\underline{V}/\Lambda_0$. For the sake of brevity, we let E denote $\underline{V}/\Lambda_0$ in the rest of this section.

Suppose that $X \subset E$ is an open subset of an irreducible analytic curve. Recall that X is said to be transverse, at a smooth point x of X, to the fibre $E_{\pi(x)}$, if

$$T'_x X + T'_x T' E_{\pi(x)} = T'_x E \quad ;$$

equivalently, $T'_x X$ does not lie in $\mathrm{Ker}\,\partial\pi_x$. Let π_X denote the restriction of π to X, and observe that X is transverse at x to $E_{\pi(x)}$ iff $\partial\pi_X(x) \neq 0$. Of course, this is equivalent to the existence of a local inverse π_X^{-1} on some neighbourhood U, of $\lambda = \pi(x)$ such that $\pi_X^{-1}(\lambda) = x$. By definition $\pi \circ \pi_X^{-1} = \mathrm{id}_U$, hence π_X^{-1} is a local section of E over U, describing the part of X over U that passes through x.

The local section π_X^{-1} generates $\tilde{\pi}_X^{-1}$, and we may lift this up into a neighbourhood, U' say, of x to obtain

$$\tilde{\pi}_X^{-1}\circ\pi : U' \longrightarrow H^0(Q(K),\mathcal{O}(E)) \quad .$$

Definition 2.16 For $x \in U'$, the section $\tilde{\pi}_X^{-1}\circ\pi(x)$, is said to osculate X at x, and is called the osculating section at x.

Remark 2.17 In the notation above, the global section $\tilde{\pi}_X^{-1} \circ \pi(x)$ is written $(\tilde{\pi}_X^{-1})_{\pi(x)}$.

Now, the information that determines $\tilde{\pi}_X^{-1} \circ \pi(x)$ is simply the 2nd order jet of π_X^{-1} at $\pi(x)$, which is completely specified by X. Hence the local expressions $\tilde{\pi}_X^{-1} \circ \pi$ patch together to give a globally defined curve on $X_* = \{x \in X \; ; \; X \text{ transverse at } x \text{ to } E_{\pi(x)}\}$.

Remark 2.18 $X_* \subset X \backslash X_{sing}$, and provided that X is not vertical, the difference is at most a set of isolated points. Since X is irreducible, $X \backslash X_{sing}$ is connected and hence X_* is connected. Note that if X is vertical then X_* is empty.

Theorem 2.19 For X an open subset of an irreducible analytic curve in E, there exists a canonical null holomorphic curve $w_X : X_* \longrightarrow V$, given locally by $w_X = \tau^{-1} \circ \tilde{\pi}_{X_*}^{-1} \circ \pi_{X_*}$. If X does not lie on a global holomorphic section, then the Gauss map γ_X is simply the projection π_{X_*}.

Proof Holomorphicity and nullity follow immediately from 2.13 and 2.15 respectively.

$$\partial w_X = \tau^{-1} \circ \partial\tilde{\pi}_{X_*}^{-1} \circ \partial\pi_{X_*} \qquad \text{implies that}$$

$$\gamma_X(x) = [\partial w_X(x)] = [\tau^{-1} \circ \partial\tilde{\pi}_{X_*}^{-1}(T'_{\pi(x)}Q(K)] ,$$

and from 2.15 this is $\pi(x)$.

Remark 2.20 If x is a singular point of X such that X reduces at x to the union of smooth curves, then the transversality condition can be checked for each component of X at x seperately. Therefore it may be possible to extend w_X over such points by lifting it to a desingularization of X at x.(We will return to this point later, but note that this phenomenon may occur in the following.)

Suppose that M is a Riemann surface and that $\Gamma : M \longrightarrow E$ is a holomorphic curve such that $\gamma = \pi \circ \Gamma$ is non-constant. $\Gamma(M)_*$ is a connected smooth curve in E, from 2.19 there exists the null curve $w_\Gamma : \Gamma(M)_* \longrightarrow V$.

Definition 2.21 For $\Gamma : M \longrightarrow E$, let os(Γ) ; $\Gamma^{-1}(\Gamma(M)_*) \longrightarrow V$ denote the curve $w_\Gamma \circ \Gamma$.

Recall that Γ is said to be transverse at $\xi \in M$, to the fibre $E_{\gamma(\xi)}$, if $\Gamma(W)$ is transverse at $\Gamma(\xi)$, for some sufficiently small neighbourhood W of ξ. Let

$$M^* = \{\xi \in M \ ; \ \Gamma \text{ transverse to } E_{\Gamma(\xi)} \text{ at } \Gamma(\xi)\}$$

and note that

$$M^* \supset \{ \xi \in M \ ; \ \partial\dot{\gamma}(\xi) \neq 0 \} .$$

Proposition 2.22 os(Γ) extends over M^*.

Proof Suppose that $\xi_0 \in M^* \setminus \Gamma^{-1}(\Gamma(M)_*)$; since Γ is transverse at ξ_0, there exists a neighbourhood W of ξ_0 such that $\Gamma(W)$ is transverse at $\Gamma(\xi_0)$. Hence $\pi^{-1}_{\Gamma(W)}$ exists on a neighbourhood U of $\zeta_0 = \gamma(\xi_0)$ and thus $\tilde{\pi}^{-1}_{\Gamma(W)}$ exists on U. But $\tilde{\pi}^{-1}_{\Gamma(W)}(\zeta) = \tilde{\pi}^{-1}_{\Gamma(M)_*}(\zeta)$ for $\zeta \in U \setminus \{\zeta_0\}$, so $\tilde{\pi}^{-1}_{\Gamma(M_*)} \circ \gamma$ extends over W, and thus $os(\Gamma)$ extends over M^*.

The following is an immediate consequence of 2.19 :

Theorem 2.23 A holomorphic curve $\Gamma : M \longrightarrow \underline{V}/\Lambda_0$ such that $\gamma = \pi \circ \Gamma$ is non-constant, may by osculation be canonically transformed into a null holomorphic curve $os(\Gamma) : M^* \longrightarrow V$. Furthermore, if the image of Γ does not lie on a global holomorphic section of $\underline{V}/\Lambda_0$, then γ is the Gauss map of $os(\Gamma)$.

(E) The Correspondence.

In §2.C we saw that any non-constant null holomorphic curve in V, possesses a spectral transform into $\underline{V}/\Lambda_0$. Conversely, in §2.D we have described how to osculate any non-vertical holomorphic curve in $\underline{V}/\Lambda_0$ to produce a null curve in V. In this section we show that these transformations are essentially inverse to each other, i.e. we establish Theorem 2.26 below. The key to this is the following :

<u>Lemma 2.24</u> If $\Omega : M \longrightarrow V$ is a null holomorphic curve such that γ_Ω is non-constant, then osculating the spectral transform of Ω gives Ω, i.e. $\mathrm{os} \circ \mathrm{sp}(\Omega) = \Omega$.

<u>Proof</u> In order to represent $\mathrm{sp}(\Omega)$ as a local section, suppose that $\gamma'(\xi_0) \neq 0$, and that γ_Ω^{-1} is a local inverse on a neighbourhood U of $\zeta_0 = \gamma_\Omega(\xi_0)$, such that $\gamma_\Omega^{-1}(\zeta_0) = \xi_0$. The part of $\mathrm{sp}(\Omega)$ that lies over U and passes through $\mathrm{sp}(\Omega)(\xi_0)$ is given on U by the local section :

$$\mathrm{sp}(\Omega) \circ \gamma_\Omega^{-1}(\zeta) \;=\; \Omega \circ \gamma_\Omega^{-1}(\zeta) \;+\; (\Lambda_0)_\zeta \quad .$$

We must calculate the 2nd order jet of this local section at ξ_0 and show that it equals $\sigma_{\Omega(\xi_0)}$: in order to do this we work in some (well chosen) local coordinates on $\underline{V}/\Lambda_0$.

If we fix $(\ ,\ ) \in K$, a choice of trivialization t of Λ over U furnishes local coordinates (ζ,η) on $\underline{V}/\Lambda_0$ over U, in which local sections are represented in the following way :

$$\text{if} \quad \sigma(\zeta) = F(\zeta) + (\Lambda_0)_\zeta \quad \text{on U, where} \quad F : U \longrightarrow V$$

then

$$\eta = f(\zeta) = (F(\zeta), t(\zeta))$$

gives σ in (ζ,η)-coordinates.

Now, suppose that $\Omega'(\xi_0) \neq 0$, and hence that $\Omega' \circ \gamma_\Omega^{-1}$ trivializes Λ on some neighbourhood $U' \subset U$, of ζ_0 ; consequently we get the following local coordinate representation of $\mathrm{sp}(\Omega) \circ \gamma_\Omega^{-1}$ on U' :

$$\eta = f(\zeta) = (\Omega \circ \gamma_\Omega^{-1}(\zeta), \Omega' \circ \gamma_\Omega^{-1}(\zeta)) \quad .$$

Now, since $\Omega'(\xi_0)$ and $\Omega''(\xi_0)$ lies in $\gamma_\Omega(\xi_0)_0$, we have

$$(\Omega(\xi),\Omega'(\xi)) = (\Omega(\xi_0) + (\xi-\xi_0)\Omega'(\xi_0) + \dots ,\Omega'(\xi_0)+ (\xi-\xi_0)\Omega''(\xi_0) + \dots)$$

$$= (\Omega(\xi_0),\Omega'(\xi)) + O[(\xi-\xi_0)^3] \quad .$$

Also, note that

$$\gamma_\Omega^{-1}(\zeta) = \gamma_\Omega^{-1}(\zeta_0) + \frac{d\gamma_\Omega^{-1}}{d\zeta}(\zeta_0)(\zeta-\zeta_0) + \dots$$

$$= \xi_0 + h(\zeta)(\zeta-\zeta_0)$$

say, and hence $(\xi-\xi_0)^3 \sim (\zeta-\zeta_0)^3$. Consequently

$$f(\zeta) = (\Omega(\xi_0),\Omega' o\, \gamma_\Omega^{-1}(\zeta)) + O[(\zeta-\zeta_0)^3]$$

$$= \sigma_{\Omega(\xi_0)}(\zeta) + O[(\zeta-\zeta_0)^3] \quad .$$

Hence $os(sp(\Omega))(\xi_0) = \Omega(\xi_0)$ for $\xi_0 \in M\setminus(B_\Omega \cup B_{\gamma_\Omega})$, therefore $os(sp(\Omega))$ extends analytically over M and equals Ω.

<u>Lemma 2.25</u> Suppose that $\Gamma_1,\Gamma_2 : M \longrightarrow \underline{V}/\Lambda_0$ are holomorphic curves whose images do not lie on a global holomorphic section or in a fibre. Then $os(\Gamma_1) = os(\Gamma_2)$ implies that $\Gamma_1 = \Gamma_2$.

<u>Proof</u> From 2.23 the projections $\gamma_1 = \pi \circ \Gamma_1$ and $\gamma_2 = \pi \circ \Gamma_2$ are the Gauss maps of $os(\Gamma_1)$ and $os(\Gamma_2)$ respectively ; so if $os(\Gamma_1) = os(\Gamma_2)$ then $\gamma_1 = \gamma_2 = \gamma$ say. Suppose that $\gamma'(\xi_0) \neq 0$ and that γ^{-1} is a local inverse on some neighbourhood U of $\zeta_0 = \gamma(\xi_0)$ such that $\gamma^{-1}(\zeta_0) = \xi_0$. $\Gamma_1 \circ \gamma^{-1}$ and $\Gamma_2 \circ \gamma^{-1}$ are local sections of $\underline{V}/\Lambda_0$ over U describing Γ_1 and Γ_2 respectively. Thus

$$\mathrm{os}(\Gamma_i)(\xi) = \tau^{-1} \circ (\tilde{\Gamma_i \circ \gamma})_{\gamma(\xi)} \qquad \text{for} \quad i = 1,2, \ \xi \in U \ ,$$

and hence

$$(\tilde{\Gamma_1 \circ \gamma^{-1}})_{\gamma(\xi)} = (\tilde{\Gamma_2 \circ \gamma^{-1}})_{\gamma(\xi)} \quad .$$

Comparing the constant terms in this equation at ξ_0 gives

$$\Gamma_1 \circ \gamma^{-1}(\gamma(\xi_0)) = \Gamma_2 \circ \gamma^{-1}(\gamma(\xi_0))$$

$$\text{i.e.} \qquad \Gamma_1(\xi_0) = \Gamma_2(\xi_0) \quad .$$

So $\Gamma_1 = \Gamma_2$ on an open subset of M and hence they agree on M.

Let $C(M,\underline{V}/\Lambda_0)$ be the set of holomorphic curves $\Omega : M \longrightarrow \underline{V}/\Lambda_0$ whose image does not lie on a global holomorphic section or in a fibre , and let N(M,V) be the collection of null holomorphic curves whose image does not lie on an affine null line on V. To summarize the above, we have :

Theorem 2.26 (Weierstrass - Hitchin Correspondence) Let M be a Riemann surface. There exists the following canonical correspondence :

$$N(M,V) \underset{\mathrm{os}}{\overset{\mathrm{sp}}{\rightleftarrows}} C(M,\underline{V}/\Lambda_0) \quad .$$

If the image of a non-constant curve $\Gamma : M \longrightarrow \underline{V}/\Lambda_0$ lies on a global holomorphic section σ_v, say, then $\mathrm{os}(\Gamma)(\xi) = v$ for all $\xi \in M$. The collection of affine null planes which pass through v corresponds under τ to σ_v ; so one might say '$\mathrm{sp}(\mathrm{os}(\Gamma)) = \sigma_v$' but note that the parameterization is lost.

If a non-constant curve $\Omega : M \longrightarrow V$ lies on an affine null line (so γ_Ω is constant), it is of the form $\Omega(\xi) = v + h(\xi)\nu$ where $v \in V$, $\nu \in K\setminus\{0\}$ and $h \in \mathcal{O}_M$. Thus

$$\begin{aligned} sp(\Omega)(\xi) &= \Omega(\xi) + \gamma_\Omega(\xi)_0 = v + h(\xi)\nu + [\nu]_0 \\ &= v + [\nu]_0 \quad . \end{aligned}$$

We observed in 2.7 that the collection of global holomorphic sections which pass through $v + [\nu]_0$ in $\underline{V}/\Lambda_0$, corresponds to the affine plane $v + [\nu]_0$ in V, which contains $\Omega(M)$: again there is a 'partial correspondence'.

Remark 2.27 A curve in $H^0(\mathbb{P},\mathcal{O}(E))$ generated by a local section σ of E, is determined by the equation

$$\sigma(\zeta) = \tilde{\sigma}_{\zeta_0}(\zeta) + O[(\zeta-\zeta_0)^{d+1}]$$

and consequently, $\frac{d\tilde{\sigma}_{\zeta_0}}{d\zeta_0}$ has a single root of order d at ζ_0. As we have seen above, when $\deg(E) = 2$ this amounts to the nullity constraint on $\tilde{\sigma}$. If $\deg(E) = 1$ then, since every global holomorphic section of E has a single root, there is no constraint on $\tilde{\sigma}$: this case may in fact be viewed as the classical duality between curves in $\mathbb{P}_2$ and $\mathbb{P}_2^*$.

It might be interesting to understand the correspondence for $d \geqslant 3$ but this is not at all clear, for the following reason : we saw above that $\deg(E) = 2$ has a natural interpretation in terms of the geometry in $H^0(\mathbb{P},\mathcal{O}(E))$; it is difficult to imagine a similar interpretation for $\deg(E) \geqslant 3$, especially for $d \geqslant 5$.

Perhaps this point is clearer in local coordinates : a choice of local coordinates on E over $\mathbb{P}\setminus\{\mu\}$ furnishes an isomorphism $H^0(\mathbb{P},\mathcal{O}(E)) \simeq C^{d+1}$ and for $\deg(E) = 2$, the global section $a + b\zeta + c\zeta^2$ has equal roots iff the discriminant vanishes. Hence observe that the conformal structure is represented by the descriminant ; how would one interpret this for $d = 3,4$ and $d \geqslant 5$?

It is perhaps more reasonable to expect interesting generalizations of the Weierstrass-Hitchin correspondence in the context of Einstein-Weyl geometries.

§3 The Weierstrass Representation Formulae in Free Form.

(A) Introduction.

In §3.B we consider the intertwining of the action of the group of cone preserving transformations of V on N(M,V) with the group of holomorphic bundle automorphisms on $C(M,\underline{V}/\Lambda_0)$, see 3.4. In 3.12 we describe this explicitly in local coordinates.

Our primary purpose in this chapter is to understand the Weierstrass representation formulae, as described in §1.C, in geometric terms via the Weierstrass - Hitchin correspondence of 2.26. This necessitates the introduction of local coordinates into the picture, i.e. we identify $Q(K) \simeq \mathbb{P}$ and fix an isomorphism $\underline{V}/\Lambda_0 \simeq \mathcal{O}(2)$, (the point being that one might equally well start with a bundle of degree 2 over $\mathbb{P}$ and develop the picture from there, see §3.C). This enables us to define a 3-dimensional representation,

$$\rho : PGL(2,C) \longrightarrow GL(H^0(\mathbb{P},\mathcal{O}(2))) \quad ,$$

which mediates in the intertwining of the effect of orthogonal base changes in $V \simeq H^0(\mathbb{P},\mathcal{O}(2))$ and changes in choice of coordinate on $\mathbb{P} \simeq Q(K)$, and consequently appears in the formulae of 3.12.

We determine necessary and sufficient conditions for the existence of a Weierstrass representation in free form for a null curve Ω on a neighbourhood of a point ξ_0, solely in terms of the geometry of Ω at ξ_0. We close this chapter with some trivial remarks regarding the relationship between the Weierstrass formulae and J_2, the CR - structure of Eells and Salamon.

(B) A Group Isomorphism.

In this section we describe a canonical group isomorphism between cone preserving transforms of V and holomorphic bundle automorphisms of $\underline{V}/\Lambda_0$. Our first object here is to identify those $A \in GL(V)$ that preserve the conformal structure, K : such transformations map null curves to null curves.

Definition 3.1 (i) Let $O_K(V) = \{A \in GL(V) \ ; \ AK = K\}$. Elements of $O_K(V)$ are the cone preserving transforms of V.

(ii) For $E \longrightarrow M$ a holomorphic vector bundle over a complex manifold M, let B(E) denote the set of holomorphic bundle automorphisms.

Recall that for $(\ , \ ) \in K$, there exist orthogonal bases for V, giving $V \cong C^3$ such that $(z,z) = zz^T$ and hence $K \simeq \{ \ z^2 \in C^3 \ ; \ z_1^2 + z_2^2 + z_3^2 = 0 \ \}$.

Proposition 3.2 $A \in O_K(V)$ iff a matrix A say, which represents A with respect to an orthogonal basis for V, satisfies $\lambda A \in O(3,C)$ for some $\lambda \in C^*$. Consequently, $O_K(V) \simeq SO(3,C) \times C^*$.

Proof Suppose that
$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$
satisfies $zMz^T = 0$ for all $z \varepsilon K$. Since zz^T is irreducible, Nullstellensatz implies that zz^T divides zMz^T. Equating coefficients reveals that this implies $a = c = i$, $b = -d$, $c = -g$ and $f = -h$. Thus M is necessarily the sum of a diagonal matrix and a skew-symmetric matrix. If A preserves K then AA^T must be of this form, but AA^T is symmetric and hence it must be diagonal : thus $\lambda A \varepsilon O(3,C)$ for some $\lambda \varepsilon C^*$.

Any $A \varepsilon O_K(V)$ induces a biholomorphism $\alpha : Q(K) \longrightarrow Q(K)$, where $\alpha(\lambda) = A\lambda$, and is actually determined up to a scalar by this induced map : for, suppose that $A,B \varepsilon O_K(V)$ induces the same element of $Aut(Q(K))$, then $Av = \lambda Bv$ for $v \varepsilon K$, where $\lambda : Q(K) \longrightarrow C^*$ is holomorphic and therefore constant, thus $A = \lambda B$ on K and hence $A = \lambda B$ on V. Further, if $\lambda \varepsilon Q(K)$ then $(A\lambda)_0 = A\lambda_0$: for, suppose that $\mu \subset A\lambda_0$ is null, then $A^{-1}\mu$, which lies on λ_0, is null and hence $A^{-1}\mu = \lambda$, thus $A\lambda_0 \cap K = A\lambda$. Consequently A maps affine null planes to affine null planes and we have the following commutative diagram :

$$\begin{tikzcd} O_K(V) \arrow[rr, "\beta"] \arrow[dr] & & B(\underline{V}/\Lambda_0) \arrow[dl] \\ & Aut(Q(K)) & \end{tikzcd}$$

where $A^{\beta}(\pi) = A\pi$ and the vertical arrows designate the induced automorphism of $Q(K)$. (Note that elements of $B(\underline{V}/\Lambda_0)$ are similarly determined up to a scaling factor by the automorphism that they induce on $Q(K)$.) Observe that β is in fact a group homomorphism and that the above is a commutative diagram of groups.

Now, $\Theta \in B(\underline{V}/\Lambda_0)$, inducing $\theta \in \mathrm{Aut}(Q(K))$, may be transformed, via the twistor transform τ, into an element of $GL(V)$ in the following way ; let $\Theta^{\tau} \in GL(V)$ be given by :

$$\Theta^{\tau}(v) = \tau^{-1}(\Theta \circ \sigma_v \circ \theta^{-1}) \quad \text{for} \quad v \in V .$$

Since for any $\eta \in H^0(Q(K), \mathcal{O}(\underline{V}/\Lambda_0))$,

$$\Theta \circ \eta(\lambda) \in Q(K) \quad \text{iff} \quad \eta(\lambda) = \lambda$$

observe that $\eta \longrightarrow \Theta \circ \eta \circ \theta^{-1}$ preserves the conformal structure $\tau(K)$ in $H^0(Q(K), \mathcal{O}(\underline{V}/\Lambda_0))$ and hence $\Theta^{\tau} \in O_K(V)$. It is clear that $\tau : B(\underline{V}/\Lambda_0) \longrightarrow O_K(V)$ is a group homomorphism ; we now show that $\tau = \beta^{-1}$, i.e. we have

<u>Proposition 3.3</u> $\beta : O_K(V) \longrightarrow B(\underline{V}/\Lambda_0)$ is a group isomorphism.

<u>Proof</u> For $A \in O_K(V)$, inducing $\alpha \in \mathrm{Aut}(Q(K))$, we have

$$A^{\beta\tau}(V) = \tau^{-1}(A^{\beta} \circ \sigma_v \circ \alpha^{-1}) .$$

But, $$A^{\beta} \circ \sigma_v \circ \alpha^{-1}(\lambda) = A^{\beta} \circ \sigma_v(A^{-1}\lambda) = A^{\beta}(v + (A^{-1}\lambda)_0)$$

$$= Av + \lambda_0 = \sigma_{Av}(\lambda) ,$$

hence $\tau^{-1}(A^{\beta} \circ \sigma_v \circ \alpha^{-1}) = Av$, and thus

$$\beta\tau = id_{O_K(V)} \quad .$$

We now show that Ker(τ) is trivial. Suppose that $\Theta \in B(\underline{V}/\Lambda_0)$ satisfies $\Theta^{\tau} = id_V$, so

$$\Theta \circ \eta \circ \theta^{-1} = \eta \quad \text{for all} \quad \eta \in H^0(Q(K), \mathcal{O}(\underline{V}/\Lambda_0)) \quad ,$$

or equivalently,

$$\Theta \circ \eta = \eta \circ \theta \quad .$$

For $\lambda \in Q(K)$, let $\eta \in \tau(K)$ have a double root at λ, then

$$\Theta \circ \eta(\lambda) = \Theta(\lambda) = \theta(\lambda) \in Q(K) \ ,$$

but $\quad \eta \circ \theta(\lambda) \in Q(K) \quad \text{iff} \quad \theta(\lambda) = \lambda$.

Since this holds for every $\eta \in \tau(K)$, we have $\theta = id_{Q(K)}$. Thus $\Theta \circ \eta = \eta$ for every $\eta \in H^0(Q(K), \mathcal{O}(\underline{V}/\Lambda_0))$ and hence $\Theta = id_{\underline{V}/\Lambda_0}$. So Ker($\tau$) $= id_{\underline{V}/\Lambda_0}$. We saw above that $\beta\tau = id_{O_K(V)}$, consequently for $\Theta \in B(\underline{V}/\Lambda_0)$,

$$\Theta^{\tau\beta\tau} = \Theta^{\tau} \quad ,$$

thus $\Theta^{\tau\beta} = \Theta$ and hence $\tau = \beta^{-1}$.

We now compare the effect of a cone preserving transform on a null curve with the corresponding transform on its spectral curve.

<u>Proposition 3.4</u> (i) The following diagram commutes :

$$\begin{array}{ccc} N(M,V) & \xrightarrow{\quad A \quad} & N(M,V) \\ {\scriptstyle sp}\downarrow & & \downarrow{\scriptstyle sp} \\ C(M,\underline{V}/\Lambda_0) & \xrightarrow{\quad A^{\beta} \quad} & C(M,\underline{V}/\Lambda_0) \end{array}$$

where A and A^{β} denote the obvious maps induced by $A \in O_K(V)$.

(ii) Correspondingly, for $\Theta \in B(\underline{V}/\Lambda_0)$ and $\psi \in C(M,\underline{V}/\Lambda_0)$ we have,

$$os(\Theta\psi) = \Theta^{\tau} os(\psi) \quad .$$

<u>Proof</u> For $\Omega \in N(M,V)$ and $\xi \in M$ such that $\Omega'(\xi) \neq 0$, we have

$$sp(A\Omega)(\xi) = A\Omega(\xi) + (A\Omega'(\xi))_0 \quad ,$$

but $$(A\Omega'(\xi))_0 = A(\Omega'(\xi))_0 \quad ,$$ hence

$$sp(A\Omega) = A^{\beta} sp(\Omega)$$

on an open subset of M, and hence on M.

(ii) follows immediately from (i) and 2.26.

(C) Local Coordinate Transforms and the Adjoint Representation of PGL(2,C).

We saw in §2 that a conformal structure K, on a 3-dimensional complex vector space V, arises canonically (via τ) as the cone in $H^0(Q(K),\mathcal{O}(\underline{V}/\Lambda_0))$ of sections which possess a double root. The results of 3.B, together with the necessity to introduce local coordinates, suggest that a shift in viewpoint, which emphasizes the picture from the base in $\underline{V}/\Lambda_0$, should be our next step. (In particular, we will see that orthogonal bases in V arise naturally from appropriate choices of local coordinate on the base.)

Let $\mathbb{P}$ be the projective space of lines in a 2-dimensional complex vector space and suppose that $E \longrightarrow \mathbb{P}$ is a holomorphic line bundle of degree 2 ; for the sake of brevity let H^0 denote $H^0(\mathbb{P},\mathcal{O}(E))$. Observe that H^0 possesses a canonical conformal structure K_E, given by the cone of sections which possess a double root and furthermore, that there exists the following embedding of $\mathbb{P}$ in $\mathbb{P}(H^0)$:

$$q : \mathbb{P} \xrightarrow{\sim} Q(K_E)$$

where $q(p)$ = line of sections in H^0, vanishing with multiplicity 2 at p.

<u>Proposition 3.4</u> E may be canonically identified with the collection of affine null planes in H^0.

Proof The collection of affine null planes in H^0 is simply

$$\underline{H}^0/\Lambda_0 \longrightarrow Q(K_E) \quad ,$$

we exhibit a bundle isomorphism $E \longrightarrow q^{-1}\underline{H}^0/\Lambda_0$. Following 2.7(i), for $e \varepsilon E$ such that $\pi(e) = p$, let

$$\Pi_e = \{ \sigma \varepsilon H^0 \ ; \ \sigma(p) = e \} \ .$$

Clearly $\Pi_e = e + \Pi_p$ and Π_e is an affine plane in H^0. Furthermore, observe that

$$\Pi_p \cap K_E = q(p) \quad ,$$

in particular Π_p intersects K_E in a single line and is therefore null. $e \longrightarrow \Pi_e$ is linear on fibres and $\mathrm{Ker}_p(\Pi) = 0$ for all $p \varepsilon \mathbb{P}$, thus Π is a bundle isomorphism.

Remark 3.5 From 2.6 we have the twistor correspondence

$$\tau : H^0(\mathbb{P},\mathcal{O}(E)) \longrightarrow H^0(Q(K_E),\mathcal{O}(\underline{H}\ /\Lambda_0)) \ .$$

In view of 3.4 we ought to view the twistor isomorphism in this context, as arising from the identification $E \simeq q^{-1}\underline{H}^0/\Lambda_0$.

Now, recall that $\mathbb{P}$ has a unique projective structure, i.e. there exists a unique maximal atlas for $\mathbb{P}$ in which the transition functions lie in the group of fractional linear transforms, PGL(2,C) : any such chart may be extended over $\mathbb{P}$ minus 1 point, (see [Gu] for information regarding projective structures on Riemann surfaces).

Definition 3.6 A local coordinate ζ, which lies in the projective structure of $\mathbb{P}$, is called a local affine coordinate (l.a.c.) on $\mathbb{P}$.

In order to specify a l.a.c. ζ on $\mathbb{P}$ we must choose some 'point at infinity', $\zeta_\infty \in \mathbb{P}$, and a scaling factor when identifying $\mathbb{P}\setminus\{\zeta_\infty\} \xrightarrow{\sim} C$. Now, a trivialization of E over $\mathbb{P}\setminus\{\zeta_\infty\}$ is simply a global holomorphic section of E lying on that line in K_E of sections which have a double root at ζ_∞ : accordingly, the collection of such 'projective trivializations' of E may be identified with K_E. Recall that the Picard group of $\mathbb{P}$ is generated by the hyperplane bundle $O(1) \longrightarrow \mathbb{P}$ and consequently $E \simeq O(2)$, see [G-H]. The collection of isomorphisms $E \simeq O(2)$ is simply C^*, corresponding to a choice of scale in the fibres of E. The choice of this scale has the effect of coupling the l.a.c.'s to the projective trivializations of E ; for, thinking of $O(2)$ as $T'\mathbb{P}$, observe that a choice of l.a.c. ζ provides the projective trivialization $\frac{d}{d\zeta}$ of $T'\mathbb{P}$ and conversely, given a projective trivialization t over $\mathbb{P}$, integrating the equation $\frac{d}{d\zeta} = t$ yields a uniquely specified l.a.c. on $\mathbb{P}$.

The expression for an element of $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$ with respect to a l.a.c. ζ, can be at most quadratic in ζ, hence $\left\{\frac{d}{d\zeta}, \zeta\frac{d}{d\zeta}, \zeta^2\frac{d}{d\zeta}\right\}$ gives a basis for $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$.

The embedding

$$q : \mathbb{P} \longrightarrow Q(K_{T'\mathbb{P}}) \hookrightarrow \mathbb{P}(H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P})))$$

is given by $q(\zeta_0) = \left[\alpha(\zeta-\zeta_0)^2\frac{d}{d\zeta}\right]$, which, with respect to the above basis, gives $q(\zeta_0) = [1,-2\zeta_0,\zeta_0^2]$.

Consequently, a choice of (l.a.c.)ζ yields an isomorphism $Q(K_{T'\mathbb{P}}) \simeq \tilde{Q}_1$, where

$$\tilde{Q}_1 = \{ [a,b,c] \in \mathbb{P}_2 \; ; \; b^2 - 4ac = 0 \} \quad .$$

A choice of (l.a.c.)ζ yields the following diagram :

$$\begin{tikzcd}
H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P})) \arrow[r, "\tau"] \arrow[d] & H^0(Q(K_{T'\mathbb{P}}),\mathcal{O}(\underline{H}^0/\Lambda_0)) \arrow[d] \\
C^3 \arrow[r, "\tau"] & H^0(\tilde{Q}_1,\mathcal{O}(\underline{C}^3/\Lambda_0))
\end{tikzcd}$$

The horizontal isomorphisms are, of course, canonical, the vertical isomorphisms are achieved simultaneously by the choice of (l.a.c.)ζ. Viewing the top isomorphism in the light of 3.5, this diagram provides a natural explanation of the fact that $T'\mathbb{P}$ may be viewed as the collection of affine null planes in C^3, where $(a,b,c) \in C^3$ corresponds to the global holomorphic section of null affine planes passing through it, given by $(a + b\zeta + c\zeta^2)\frac{d}{d\zeta}$.

Now observe that, if given (l.a.c.)ζ we choose the following basis for $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$:

$$-\frac{1}{2}(1-\zeta^2)\frac{d}{d\zeta} \quad , \quad -\frac{i}{2}(1+\zeta^2)\frac{d}{d\zeta} \quad , \quad -\zeta\frac{d}{d\zeta} \quad ,$$

then solving,

$$\zeta^2 - 2\zeta_0\zeta + \zeta_0^2 = -\frac{A}{2}(1-\zeta^2) - \frac{iB}{2}(1+\zeta^2) - C\zeta$$

gives

$$A = 1-\zeta_0^2 \quad , \quad B = i(1+\zeta_0^2) \ , \quad C = 2\zeta_0 \quad .$$

Thus $\quad q : \mathbb{P} \longrightarrow Q(K_{T'\mathbb{P}})$ becomes

$$q(\zeta) = [1-\zeta^2,\ i(1+\zeta^2),\ 2\zeta] \ ,$$

and we have $Q(K_{T'\mathbb{P}}) \simeq Q_1$.

In the corresponding diagram, having obtained this orthogonal basis for $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$, we find that $z = (z_1,z_2,z_3) \in C^3$ is represented in H^0 by

$$\sigma_z(\zeta) = -\left\{\left(\frac{z_1+iz_2}{2}\right) + z_3\zeta - \left(\frac{z_1-iz_2}{2}\right)\zeta^2\right\}\frac{d}{d\zeta} \quad .$$

Since a bundle automorphism of E is determined up to a scalar by the induced map on $\mathbb{P}$, observe that a choice of isomorphism $E \simeq T'\mathbb{P}$ has the effect of representating each projective equivalence class of B(E), by a unique element of the form $\partial\theta$, where $\theta \in PGL(2,C)$; (note in particular, that the pure dilations of E, i.e. fibre rescalings, are all represented simply by the identity map on $\mathbb{P}$). We now examine the representation ρ, of PGL(2,C) on $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$ given by $\rho(\theta) = (\partial\theta)^T$. (This corresponds to the coordinate transformation induced on $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$ when we change l.a.c. on $\mathbb{P}$, $\omega = \theta(\zeta)$.)

Suppose that,

$$\theta(\zeta) = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \quad \text{where} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,C) \ ,$$

then

$$\theta^{-1}(\zeta) = \frac{\delta\zeta - \beta}{-\gamma\zeta + \alpha} \quad \text{and} \quad \frac{d\theta}{d\zeta}(\zeta) = (\gamma\zeta + \delta)^{-2} \ ,$$

so

$$\frac{d\theta}{d\zeta}(\theta^{-1}(\zeta)) = (\gamma\zeta - \alpha)^2 \ .$$

Consequently for $\eta = (a + b\zeta + c\zeta^2)\frac{d}{d\zeta} \in H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$, we have

$$\begin{aligned}
((\partial\theta)^T\eta)(\zeta) &= \partial\theta \circ \eta \circ \theta^{-1}(\zeta) \\
&= \left\{a + b\left(\frac{\delta\zeta - \beta}{-\gamma\zeta + \alpha}\right) + c\left(\frac{\delta\zeta - \beta}{-\gamma\zeta + \alpha}\right)^2\right\}(\gamma\zeta-\alpha)^2\frac{d}{d\zeta} \\
&= \{a(\gamma\zeta-\alpha)^2 + b(\delta\zeta-\beta)(-\gamma\zeta+\alpha) + c(\delta\zeta-\beta)^2\}\frac{d}{d\zeta} \\
&= (a' + b'\zeta + c'\zeta^2)\frac{d}{d\zeta}
\end{aligned}$$

where,

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} \alpha^2 & -\alpha\beta & \beta^2 \\ -2\alpha\gamma & \alpha\delta+\beta\gamma & -2\beta\delta \\ \gamma^2 & -\gamma\delta & \delta^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We know from 3.3 that this matrix $\rho(A)$, preserves the conformal structure, i.e. if $b^2-4ac = 0$ then $(b')^2-4a'c' = 0$, one can verify this directly.

Remark 3.7 Recall that if $\omega = \theta(\zeta)$ then $(\zeta,\eta) \longmapsto (\zeta, \eta\frac{d}{d\zeta})$ and $(\omega,\mu) \longmapsto (\omega, \mu\frac{d}{d\omega})$ represent the same point of $T'\mathbb{P}$ iff $\mu = \eta\frac{d\theta}{d\zeta}(\zeta)$; consequently, $\{a' + b'\omega + c'\omega^2\}\frac{d}{d\omega}$ represents the section η in ω-coordinates.

Proposition 3.8 The representation $\rho : SL(2,C) \longrightarrow GL(H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P})))$ may be viewed as the adjoint representation $Ad : SL(2,C) \longrightarrow GL(s\ell(2,C))$.

Proof Simply observe that the above equation may be rewritten, using the isomorphism

$$C^3 \longrightarrow s\ell(2,C)$$

$$(a\ b\ c) \longmapsto \begin{pmatrix} b & 2a \\ -2c & -b \end{pmatrix}$$

as follows :

$$\begin{pmatrix} b' & 2a' \\ -2c' & -b' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} b & 2a \\ -2c & -b \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} .$$

Since $SL(2,C)$ is a matrix group, the adjoint representation is simply

$$Ad(y)X = yXy^{-1} , \quad \text{see [V].}$$

Recall that

$$\sigma_1 = \begin{pmatrix} & -1 \\ -1 & \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

are (minus) the usual Pauli spin matrices and let $\sigma = (\sigma_1\, \sigma_2\, \sigma_3)$. The transformation from $(a\, b\, c)$ to $(z_1\, z_2\, z_3)$ coordinates on $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$ given by

$$a = -\left(\frac{z_1 + iz_2}{2}\right) \quad , \quad b = -z_3 \quad , \quad c = \frac{z_1 - iz_2}{2} \quad ,$$

together with the above isomorphism, is effected by :

$$z \longmapsto \sigma z^T \quad .$$

Observe that ρ, with respect to $(z_1\, z_2\, z_3)$ coordinates satisfies :

$$\sigma(\rho(A)z^T) \;=\; A(\sigma z^T)A^{-1} \quad .$$

Since $\det(\sigma z^T) = -zz^T$, we have $\rho(A) \in O(3,C)$, a calculation shows that

$$\det(\rho(A)) \;=\; \det(A)^3 \quad ,$$

consequently we have a (double covering)

$$\rho : SL(2,C) \longrightarrow SO(3,C) \quad .$$

Thus ρ is simply the natural extension of the classical quaternion representation of the rotational part of $PGL(2,C)$, i.e. we have

$$\begin{tikzcd} SU(2) \arrow[r] \arrow[d, hook] & SO(3,\mathbb{R}) \arrow[d, hook] \\ SL(2,C) \arrow[r] & SO(3,C) \end{tikzcd} \quad .$$

Remark 3.9 We can recover the full group $O_K(V)$ as follows : let

$$\rho' : GL(2,C) \longrightarrow SO(3,C) \times C^*$$

be given by

$$\sigma(\rho'(A)z^T) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (\sigma z^T) \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Note that $\rho(A) = \rho'(A)$ iff $A \in SL(2,C)$.

$\rho'(\lambda A) = \lambda^2\rho'(A)$ implies that $\det(\rho'(\lambda A)) = \lambda^6 \det(A)^3$,

consequently

$$iSL(2,C) \longrightarrow SO_-(3,C) \subset SO(3,C) \times C^* .$$

(Essentially we have thrown away our choice of scale.)

(D) The Weierstrass Representation Formulae.

In the next proposition we provide a geometric interpretation of the function f, which features in the free form version of the Weierstrass formulae ; c.f. §1.C and the Appendix of [H2].

The reformulations of the previous sections enable us to view the Weierstrass - Hitchin correspondence as follows ;

$$C(M,T'\mathbb{P}) \underset{sp}{\overset{os}{\rightleftarrows}} N(M,H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))) \xrightarrow{\sim} N(M,C^3) ,$$

where the second isomorphism results from a choice of l.a.c. on $\mathbb{P}$. Now, suppose that $\Omega \in N(M,C^3)$ is such that $\gamma'_\Omega(\xi_0) \neq 0$ and that γ_Ω^{-1} is inverse to γ_Ω on a neighbourhood U of $\zeta_0 = \gamma_\Omega(\xi_0)$ such that $\gamma_\Omega^{-1}(\zeta_0) = \xi_0$; then :

$$\eta(\zeta) = f(\zeta)\frac{d}{d\zeta} = sp(\Omega) \circ \gamma_\Omega^{-1}(\zeta)$$

gives an implicit description of $sp(\Omega) \circ \gamma_\Omega^{-1}(U)$.

<u>Definition 3.10</u> If $f(\zeta)\frac{d}{d\zeta}$ gives an implicit description of some part of $sp(\Omega)$ over $U \subset \mathbb{P}$, where ζ is a l.a.c., then f is a <u>spectral function</u> for Ω.

<u>Proposition 3.11</u> If γ_Ω^{-1} is inverse to the Gauss map γ_Ω of a null curve $\Omega : M \longrightarrow C^3$ on $U \subset \mathbb{P}$, and f is a spectral function for Ω over U then

$$\Omega_1 \circ \gamma_\Omega^{-1}(u) = \frac{1}{2}(1-u^2)f''(u) + uf'(u) - f(u)$$

$$\Omega_2 \circ \gamma_\Omega^{-1}(u) = \frac{i}{2}(1+u^2)f''(u) - iuf'(u) + if(u)$$

$$\Omega_3 \circ \gamma_\Omega^{-1}(u) = uf''(u) - f'(u) \qquad \text{for } u \in U .$$

<u>Proof</u> Let $\eta(\zeta) = f(\zeta)\frac{d}{d\zeta}$, the 2-jet of η at $u \in U$ is given by :

$$\tilde{\eta}_u(\zeta) = \{f(u) + f'(u)(\zeta-u) + \frac{f''(u)}{2}(\zeta-u)^2\}\frac{d}{d\zeta} .$$

The coefficients of $\tilde{\eta} : U \longrightarrow H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$ are, therefore :

$$a(u) = f(u) - uf'(u) + \tfrac{1}{2}u^2f''(u)$$

$$b(u) = f'(u) - uf''(u)$$

$$c(u) = \tfrac{1}{2}f''(u) .$$

Converting to $(z_1\, z_2\, z_3)$-coordinates in $H^0(\mathbb{P},\mathcal{O}(T'\mathbb{P}))$, we solve

$$-\tfrac{1}{2}(\Omega_1 + i\Omega_2) \circ \gamma_\Omega^{-1}(u) = a(u)$$

$$-\Omega_3 \circ \gamma_\Omega^{-1}(u) = b(u)$$

$$\tfrac{1}{2}(\Omega_1 + i\Omega_2) \circ \gamma_\Omega^{-1}(u) = c(u) \quad ,$$

from which the above formulae follow immediately.

Given a null curve $\Omega : U \longrightarrow C^3$ which is parameterized by its Gauss map, i.e. $\gamma_\Omega(\zeta) = \zeta$, and $A \in SO(3,C)$ inducing $\alpha \in Aut(Q_1)$, observe that $\tilde{\Omega} = A\Omega \circ \alpha^{-1}$ is parameterized by its Gauss map. In the next proposition, which follows easily from 3.4 together with the observations of §3.C, we describe the relationship between the spectral functions of Ω and $\tilde{\Omega}$.

<u>Proposition 3.12</u> Suppose that $\eta = f(\zeta)\frac{d}{d\zeta}$ and $\tilde{\eta} = \tilde{f}(\zeta)\frac{d}{d\zeta}$ describe 2 local sections of $T'\mathbb{P}$ over $U \subset \mathbb{P}$, where ζ is a l.a.c. on $\mathbb{P}$ and let $\Omega = os(\eta)$ and $\tilde{\Omega} = os(\tilde{\eta})$. Then :

(i) η and $\tilde{\eta}$ are related by an equation of the form

$$\tilde{\eta} = \partial\theta \circ \eta \circ \theta^{-1} \quad , \quad \text{for} \quad \theta \in SL(2,C)$$

iff

$$(\sigma\tilde{\Omega}^T)\circ\theta = \mathrm{Ad}(\theta)(\sigma\Omega^T) \quad .$$

(ii) More explicitly,

$$\tilde{f}(\zeta) = (\gamma\zeta-\alpha)^2 f\left(\frac{\delta\zeta-\beta}{-\gamma\zeta+\alpha}\right) \quad \text{for} \quad \theta = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \ \varepsilon\ SL(2,C)$$

iff Ω and $\tilde{\Omega}$ are related (by an element of SO(3,C))as follows :

$$\begin{pmatrix}\tilde{\Omega}_3\circ\theta & -(\tilde{\Omega}_1+i\tilde{\Omega}_2)\circ\theta\\ -(\Omega_1-i\Omega_2)\circ\theta & -\tilde{\Omega}_3\circ\theta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\Omega_3 & -(\Omega_1+i\Omega_2)\\ -(\Omega_1-i\Omega_2) & -\Omega_3\end{pmatrix}\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}$$

(iii) If $\phi = \mathrm{Re}(\Omega)$ and $\tilde{\phi} = \mathrm{Re}(\tilde{\Omega})$ and $\begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix} \varepsilon\ SU(2)$,

then

$$\tilde{f}(\zeta) = (\bar{\beta}\zeta+\alpha)^2 f\left(\frac{\bar{\alpha}\zeta-\beta}{\bar{\beta}\zeta+\alpha}\right)$$

iff, ϕ and $\tilde{\phi}$ are related (by an element of SO(3,$\mathbb{R}$)) as follows :

$$\begin{pmatrix}\tilde{\phi}_3\circ\theta & -(\tilde{\phi}_1+i\tilde{\phi}_2)\circ\theta\\ -(\tilde{\phi}_1-i\tilde{\phi}_2)\circ\theta & -\tilde{\phi}_3\circ\theta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\begin{pmatrix}\phi_3 & -(\phi_1+i\phi_2)\\ -(\phi_1-i\phi_2) & -\phi_3\end{pmatrix}\begin{pmatrix}\bar{\alpha} & -\beta\\ \bar{\beta} & \alpha\end{pmatrix}$$

Remark 3.13 The relation

$$\tilde{f}(\zeta) = \frac{d\theta}{d\zeta}(\theta^{-1}(\zeta))f \circ \theta^{-1}(\zeta)$$

is simply

$$\tilde{f}(\omega) = \frac{d\theta}{d\zeta}(\zeta)f(\zeta) \quad ,$$

where $\omega = \theta(\zeta)$. $\tilde{f}(\zeta)$ is therefore, a picture of $\eta = f(\zeta)\frac{d}{d\zeta}$ in ω-coordinates, i.e. $\tilde{f}(\omega)\frac{d}{d\omega}$ describes η over $\omega = \theta(\zeta)$ in ω-coordinates.

Example 3.14 In 1.15 we saw that $\Omega : C \longrightarrow C^3$, given by

$$\Omega(\zeta) = \left(\frac{1}{2}\zeta - \frac{1}{6}\zeta^3, \ \frac{i}{2}\zeta + \frac{i}{6}\zeta^3, \ \frac{1}{2}\zeta^2\right)$$

(whose real part is Enneper's surface), has the spectral function $f(\zeta) = \frac{1}{6}\zeta^3$. For $\theta = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ we have $\omega = \theta(\zeta) = -\frac{1}{\zeta}$ and

$$\tilde{f}(\zeta) = \zeta^2 f \circ \theta^{-1}(\zeta) = \zeta^2\left(\frac{-1}{6\zeta^3}\right) = \frac{-1}{6\zeta} \ .$$

The above formula yields

$$\begin{pmatrix} \tilde{\Omega}_3 & -(\tilde{\Omega}_1 + i\tilde{\Omega}_2) \\ -(\tilde{\Omega}_1 - i\tilde{\Omega}_2) & -\tilde{\Omega}_3 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} \Omega_3 \circ \theta^{-1} & (\Omega_1 + i\Omega_2)\circ \theta^{-1} \\ -(\Omega_1 - i\Omega_2)\circ\theta^{-1} & -\Omega_3 \circ \theta^{-1} \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$= \begin{pmatrix} -\Omega_3 \circ \theta^{-1} & -(\Omega_1 - i\Omega_2)\circ\theta^{-1} \\ (\Omega_1 + i\Omega_2)\circ\theta^{-1} & \Omega_3 \circ \theta^{-1} \end{pmatrix} \quad ,$$

and hence

$$\begin{pmatrix} \tilde{\Omega}_1 \\ \tilde{\Omega}_2 \\ \tilde{\Omega}_3 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \circ \theta^{-1}$$

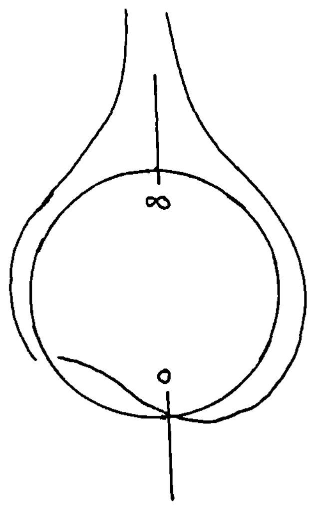
i.e.

$$\tilde{\Omega}_1(\zeta) = -\frac{1}{2}\zeta^{-1} + \frac{1}{6}\zeta^{-3}$$

$$\tilde{\Omega}_2(\zeta) = \frac{i}{2}\zeta^{-1} + \frac{i}{6}\zeta^{-3}$$

$$\tilde{\Omega}_3(\zeta) = -\frac{1}{2}\zeta^{-2}.$$

Of course $\tilde{f}(\omega) = -\frac{1}{\omega}$ gives a picture of the spectral curve of Ω over a neighbourhood of ∞ in $\mathbb{P}$, in particular the diagram on the right illustrates the global structure in $T'\mathbb{P}$ of the spectral curve of Enneper's surface.



Remark 3.15 One might use these ideas to study null curves (minimal surfaces) whose image is invariant under symmetries of C^3 ($\mathbb{R}^3$). Suppose $\Omega : C \longrightarrow C^3$ is null and has the spectral function f ; observe that $\Omega(C)$ is invariant under $\Gamma < SO(3,C)$ iff $A\Omega \circ \alpha^{-1} = \Omega$ for all $A \in \Gamma$ and hence iff

$$f(\zeta) = (\gamma\zeta - \alpha)^2 f\left(\frac{\delta\zeta - \beta}{-\gamma\zeta + \alpha}\right) \quad ,$$

for all $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ lying in the double cover of Γ, $\tilde{\Gamma}$ in SL(2,C). Note in particular that if Ω is defined on the upper $\frac{1}{2}$-plane in C, and $\tilde{\Gamma}$ is a discrete subgroup of $SL(2,\mathbb{R})$, then the above equation asserts that f is an automorphic function of weight -2.

We turn now to the question of the existence of a Weierstrass representation in free form. We have seen that $\gamma_\Omega'(\xi_0) \neq 0$ ensures the existence of a neighbourhood U of ξ_0, together with a holomorphic function f on $\gamma_\Omega(U)$ such that

$$\Omega(\xi) = \mathrm{WRFF}(f \circ \gamma_\Omega(\xi)) \qquad \text{for} \quad \xi \in U \quad .$$

In the light of 3.11 it is easy to see that such on f exists iff $sp(\Omega)$ is transverse to the fibre at $sp(\Omega)(\xi_0)$; we will show that γ_Ω may branch at ξ_0, provided that this results purely from ramification of Ω at ξ_0. (The notion of ramification is defined in Appendix A, which should be consulted if any of the notation used in the following is unfamiliar.)

<u>Proposition 3.16</u> $\Omega \in N(M,C^3)$ has a Weierstrass representation in free form on some neighbourhood U of $\xi_0 \in M$ iff $br_{\xi_0}(\gamma_\Omega) = ram_{\xi_0}(\Omega)$.

<u>Proof</u> Suppose that Ω has a Weierstrass representation on U a neighbourhood of ξ_0. From 3.11, f is a spectral function for Ω and thus describes $\tilde{U} = sp(\Omega)(U)$, which is therefore, transverse to the fibres. Hence, from 2.19 and 2.26, we can factor $\Omega|_U$ through the

spectral curve :

$$\Omega(\xi) \quad = \quad \tilde{\Omega} \circ sp(\Omega) \qquad \text{for} \quad \xi \varepsilon U \qquad (1)$$

where $\tilde{\Omega} = w_{sp(\Omega)}$ (in the notation of 2.19). Hence

$$\gamma_{\Omega}(\xi) \quad = \quad \gamma_{\tilde{\Omega}} \circ sp(\Omega)(\xi) \ , \qquad \text{for} \quad \xi \varepsilon U \ . \qquad (2)$$

Again from 2.19, we have $\gamma_{\tilde{\Omega}} = \pi|_{\tilde{U}}$, which is not branched and hence $\tilde{\Omega}$ is not ramified at $\tilde{\xi}_0 = sp(\Omega)(\xi_0)$. Thus viewing $sp(\Omega)|_U$ as a map $U \longrightarrow \tilde{U}$, we have from (1) :

$$\text{ram}_{\xi_0}(\Omega) \quad = \quad \text{br}_{\xi_0}(sp(\Omega))$$

and from (2),

$$\text{br}_{\xi_0}(\gamma_{\Omega}) \quad = \quad \text{br}_{\xi_0}(sp(\Omega))$$

hence

$$\text{br}_{\xi_0}(\gamma_{\Omega}) \quad = \quad \text{ram}_{\xi_0}(\Omega) \quad .$$

Conversely, suppose that $\text{br}_{\xi_0}(\gamma_{\Omega}) = \text{ram}_{\xi_0}(\Omega)$. Since γ_{Ω} is non-constant, there exists a neighbourhood U of ξ_0 such that $\gamma'_{\Omega}(\xi) \neq 0$ for $\xi \varepsilon U \backslash \{\xi_0\}$. From 2.19 and 2.26 there exists

$$\tilde{\Omega} : \tilde{U} \backslash \{\tilde{\xi}_0\} \longrightarrow C^3$$

such that

$$\Omega(\xi) \quad = \quad \tilde{\Omega} \circ sp(\Omega)(\xi) \qquad \text{for} \quad \xi \varepsilon U \backslash \{\xi_0\} \qquad (3).$$

Now, since $\Omega = \Omega_1 \circ \rho$ iff $sp(\Omega) = sp(\Omega_1) \circ \rho$, we have from Proposition 6 of Appendix A, that

$$\mathrm{br}_{\xi_0}(\mathrm{sp}(\Omega)) + 1 = (\mathrm{Br}_{\xi_0}(\mathrm{sp}(\Omega)) + 1)(\mathrm{ram}_{\xi_0}(\mathrm{sp}(\Omega)) + 1)$$
$$= (\mathrm{Br}_{\xi_0}(\mathrm{sp}(\Omega)) + 1)(\mathrm{ram}_{\xi_0}(\Omega) + 1)$$
$$= \mathrm{Br}_{\xi_0}(\mathrm{sp}(\Omega) + 1)(\mathrm{br}_{\xi_0}(\gamma_\Omega) + 1) \quad .$$

But, $\gamma_\Omega = \pi \circ \mathrm{sp}(\Omega)$ implies that $\mathrm{br}(\gamma_\Omega) \geqslant \mathrm{br}\,\mathrm{sp}(\Omega)$, hence $\mathrm{Br}_{\xi_0}(\mathrm{sp}(\Omega)) = 0$ and $\tilde{U}$ is biholomorphic to a disc. Thus $\tilde{\Omega}$ is an analytic function defined on a punctured disc and from (3), the boundedness of Ω at ξ_0 implies that $\tilde{\Omega}$ extends over $\tilde{U}$. (3) implies that

$$\gamma_\Omega = \gamma_{\tilde{\Omega}} \circ \mathrm{sp}(\Omega) \;,$$

and since

$$\mathrm{br}(\gamma_\Omega) = \mathrm{br}(\mathrm{sp}(\Omega))$$

we have

$$\mathrm{br}_{\tilde{\xi}_0}(\gamma_{\tilde{\Omega}}) = 0 \;,$$

thus $\gamma_{\tilde{\Omega}}^{-1}$ exists on $\gamma_\Omega(U)$ and $\gamma_{\tilde{\Omega}}^{-1} \circ \gamma_\Omega(\xi) = \mathrm{sp}(\Omega)(\xi)$, thus

$$\Omega(\xi) = \mathrm{WRFF}(\gamma_{\tilde{\Omega}}^{-1} \circ \gamma_\Omega(\xi)) \;.$$

(E) The Weierstrass Formulae and J_2.

In [E&S], Eells and Salamon describe a CR-structure, J_2, on the unit tangent sphere bundle of an oriented 3-dimensional Riemannian manifold N, with respect to which a non-constant branched immersion $\phi : M \longrightarrow N$, where M is a Riemann surface, is conformal and harmonic iff the

(Gauss) lift of ϕ to $S(N)$, $\tilde{\phi}$, is J_2-holomorphic. For $\phi : M \longrightarrow \mathbb{R}^3$, $\tilde{\phi} = (\gamma_\phi, \phi)$ and the (1,0)-distribution for J_2 is given by :

$$T_2^{1,0}\underline{\mathbb{R}}^3 = \pi_1^{-1}T^{0,1}S^2 + \pi_1^{-1}T^{1,0}S^2 \quad ,$$

where $\underline{\mathbb{R}}^3 = S^2 \times \mathbb{R}^3$, π_1, π_2 denote the projection maps, and S^2 is orientated with respect to the outward normal field. It is easy to see that the J_2-holomorphicity of $\tilde{\phi}$ may be interpreted as follows :

(i) γ_ϕ is holomorphic iff ϕ is harmonic

(ii) $d\phi(T^{1,0}M) \subset \pi_1^{-1}T^{1,0}S^2$ iff ϕ is conformal.

Observe that, using $D : S^2 \simeq Q_1$, we can rewrite the above as

$$T_2^{1,0}\underline{\mathbb{R}}^3 \simeq \pi_1^{-1}T^{1,0}Q_1 + \pi_1^{-1}\Lambda \quad ,$$

and let us refer to a holomorphic curve $\psi = (g,f) : M \longrightarrow Q_1 \times C^3$ constrained so that $[\partial f] = g$, as a <u>Λ-curve</u>, i.e. $\partial\psi(T^{1,0}M) \subset \pi_1^{-1}T^{1,0}Q_1 + \pi_1^{-1}\Lambda$. Locally, for ϕ harmonic, we can write $\phi = \frac{1}{2}(\Omega + \bar{\Omega})$, where Ω is holomorphic, hence $\partial\phi = \frac{1}{2}\partial\Omega$ and thus $\tilde{\phi}$ is J_2-holomorphic iff (γ_Ω, Ω) is a Λ-curve. So we have the diagram :

$$\begin{tikzcd}
\text{Null Curves} \arrow[r, leftrightarrow] \arrow[d, "\mathrm{Re}"'] & \Lambda\text{-curves} \arrow[d, "\mathrm{Re}"'] \\
\text{Minimal Surfaces} \arrow[r, leftrightarrow] & J_2\text{-curves}
\end{tikzcd}$$

Given a Λ-curve $\psi : M \longrightarrow Q_1 \times C^3$, composing with the projections

$$\begin{tikzcd}[column sep=small]
 & Q_1 \times C^3 \arrow[dl, "\mathrm{Re}"'] \arrow[dr] & \\
Q_1 \times \mathbb{R}^3 & & \underline{C}^3/\Lambda_0
\end{tikzcd}$$

we obtain on the left, a J_2-holomorphic curve, and on the right, a curve whose implicit description over Q_1 is a spectral function of the projection of ψ into C^3.

§4 Algebraic Minimal Surfaces.

(A) Introduction

Recall that in §2 we saw how to transform a null curve in V into a curve on $\underline{V}/\Lambda_0$, see 2.26. Having removed the constraint of nullity it is natural to consider conformally compactifying, since we can do this and preserve the essential feature of the transformed curve, namely holomorphicity. Furthermore, if we compactify correctly then we preserve the property of $\underline{V}/\Lambda_0 \longrightarrow Q(K)$ that encodes nullity, i.e. the existance of a system of rational curves of self-intersection number 2. In B we briefly consider the geometry of the appropriate compactification, which is a familiar object from algebraic geometry, namely a rational ruled surface. In C we show that having compactified $\underline{V}/\Lambda_0$, sp extends over the ends of a null meromorphic curve ; we use this to describe a natural parameterization domain. This result should be viewed as a corollary of the result of Osserman [Os 2], regarding the extension of the Gauss map for complete minimal surfaces of finite total curvature, in the context of algebraic minimal surfaces, see D.

It is unlikely that many of the results of this chapter appear here in their best form. The work is quite tentative and occasionally lapses into speculation. The hope is that some of these ideas point to further interesting problems.

(B) A Conformal Compactification of the Space of Null Affine Planes.

Those readers who are unfamiliar with the following may wish to consult [G&H : §4.3], from which this discussion is derived.

Definition 4.1 For $E \longrightarrow M$, a holomorphic vector bundle over a complex manifold M, the associated projective bundle, $\mathbb{P}(E) \longrightarrow M$, is that fibre bundle over M whose fibre at $z \in M$ is the projective space $\mathbb{P}(E_z)$.

Given an open cover $\{U_\alpha\}$ of M, together with a collection of local trivializations

$$t_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times C^k \quad ,$$

observe that the induced trivialization

$$\tilde{t}_\alpha : \mathbb{P}(E)|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{P}_{k-1} \quad ,$$

endow $\mathbb{P}(E)$ with the structure of a holomorphic $\mathbb{P}_{k-1}$-bundle over M. If $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(r,C)\}$ are the transition functions associated to $\{t_\alpha\}$ then transition functions for $\mathbb{P}(E)$ are given by the composition $\tilde{g}_{\alpha\beta}$, of $g_{\alpha\beta}$ with the usual projection $GL(r,C) \longrightarrow PGL(r,C)$.

Let $\underline{C} = M \times C$, and suppose that $E \in Pic(M)$. There is the inclusion

$$\iota : E \longrightarrow \mathbb{P}(E \oplus \underline{C})$$

whose restriction to the fibre E_z is given by

$$\begin{array}{ccccc} E_z & \longrightarrow & (E \oplus \underline{C})_z & \longrightarrow & \mathbb{P}(E \oplus \underline{C})_z \\ e & \longrightarrow & (e,1) & \longrightarrow & [e,1] \end{array} .$$

Consequently, $\mathbb{P}(E \oplus \underline{C})$ is a conformal compactification of E, in which each fibre over M is completed at infinity to a projective line. Topologically, E is 'capped off' at infinity by the addition of a copy of M.

Suppose that $M = \mathbb{P}_1$, $E \simeq O(n)$ and let $S_n = \mathbb{P}(O(n) \oplus \underline{C})$. We now review some of the salient features of the global geometry of S_n. For σ a meromorphic section of $O(n)$, let $E_\sigma \subset S_n$ denote the closure of the image of the section $(\sigma,1)$ of $O(n) \oplus \underline{C}$; in particular note that E_0 is the <u>zero section</u> of S_n. The $\mathbb{P}_1$ at infinity in S_n may be seen in this context in the following way : suppose that σ is a non-zero global holomorphic section of $O(n)$, away from the zeros of σ, $(\sigma,0)$ gives a curve in S_n whose closure is independent of the choice of σ, this curve is denoted E_∞. It is straightforward to establish the following intersection formula (where C is a generic fibre of $S_n \longrightarrow \mathbb{P}$) :

$$E_0 \cdot E_0 = n \ , \quad E_0 \cdot E_\infty = 0 \ , \quad C \cdot C = 0$$

$$E_\sigma \cdot E_0 = \text{number of zeros of } \sigma$$

$$E_\sigma \cdot E_\infty = \text{number of poles of } \sigma$$

$$E_0 \cdot C = E_\sigma \cdot C = E_\infty \cdot C = 1 \quad .$$

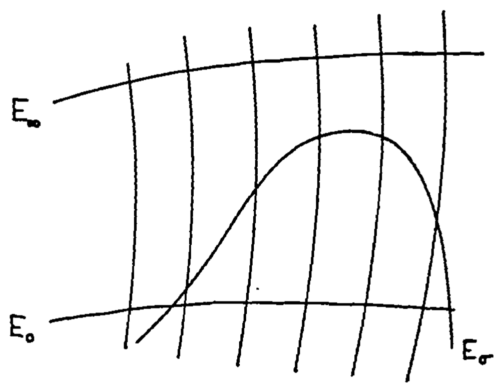
Furthermore, since the Chern class $H^1(S_n, \mathcal{O}^*) \xrightarrow{c_1} H^2(S_n, Z)$ is an isomorphism, 2 curves are linearly equivalent on S_n iff they are homologous. Consequently we can deduce, from the above formulae, that

$$E_\infty \sim E_0 - nC \quad .$$

Observe that this implies that $E_\infty \cdot E_\infty = -n$: it is easy to show that E_∞ is the unique irreducible curve on S_n having negative self-intersection number. S_n may be viewed as the $\mathbb{P}_1$-bundle over $\mathbb{P}_1$ that possesses an irreducible curve of self-intersection number $-n$. It follows from the adjunction formula that the canonical bundle K_n of S_n is given by :

$$K_n = -2E + (n-2)C \quad .$$

Now, since $\deg(\underline{V}/\Lambda_0) = 2$, $\mathbb{P}(\underline{V}/\Lambda_0 \oplus \underline{C}) \simeq S_2$ and we have $E_\infty \sim E_0 - 2C$ and $K_2 = -2E_0$. The following diagram illustrates the global structure of S_2 :



where $\sigma \in H^0(Q(K), \mathcal{O}(\underline{V}/\Lambda_0))$.

Remark 4.2 (i) S_n are rational ruled surface, in particular they are algebraic. (They embed in $\mathbb{P}_N$ as the rational normal scrolls.)

(ii) One may view S_2 as follows. For $\pi : T'\mathbb{P} \longrightarrow \mathbb{P}$, $H^0(T'\mathbb{P}, (\pi^{-1}O(2)))$ is spanned by

$$\eta\frac{d}{d\zeta} \ , \ -\frac{1}{2}(1-\zeta^2)\frac{d}{d\zeta} \ , \ -\frac{i}{2}(1+\zeta^2)\frac{d}{d\zeta} \ , \ -\zeta\frac{d}{d\zeta}$$

see Proposition 4.5 of [H3]. These sections embed $T'\mathbb{P}$ into $\mathbb{P}_3$:

$$T'\mathbb{P} \longrightarrow \mathbb{P}(H^0(T'\mathbb{P},\mathcal{O}(\pi^{-1}O(2))) \quad ,$$

where the image lies on the quadric cone :

$$\{[z_0,z_1,z_2,z_3] \ ; \ z_0 = \eta \ , \ z_1 = -\frac{1}{2}(1-\zeta^2) \ , \ z_2 = -\frac{i}{2}(1+\zeta^2) \ , \ z_3 = -\zeta\} \ .$$

This cone is a 1-point conformal compactification of $T'\mathbb{P}_1$; blowing-up the vertex $[1,0,0,0]$, we obtain S_2.

(iii) Note that E_∞ is the only curve on S_n that is distinguished canonically ; E_0, for example, is simply an irreducible curve chosen from the linear system $|E_0|$.

(iv) $T'\mathbb{P}$ may be viewed as the space of oriented lines in $\mathbb{R}^3$, for an oriented line is determined by its direction, together with the vector on it that lies closest to the origin. The twistor correspondence replaces $x \in \mathbb{R}^3$ by the global section of oriented lines which pass through x. Thus adding a $\mathbb{P}_1$ at infinity to $T'\mathbb{P}_1$ corresponds (conformally) to the addition of a point at infinity to $\mathbb{R}^3$. Hence, S_2 may be loosely described as the space of oriented geodesics of a 'flat S^3'. S_2 is homeomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ -

the space of oriented geodesics of S^3, so one might view the non-triviality of S_2 as a holomorphic product, as arising from the inability of S^3 to support a flat metric ?

(C) Null Meromorphic Curves in C^3.

The following result is in essence due to Weierstrass, see [E] :

Theorem 4.3 Suppose that $\Omega : M\setminus\{p_1, \ldots ,p_r\} \longrightarrow C^3$ is a null meromorphic curve. Compactifying $T'\mathbb{P}_1$ to S_2, as described in §4.B, $sp(\Omega)$ extends over $\{p_1, \ldots ,p_r\}$ to a holomorphic curve $sp(\Omega) : M \longrightarrow S_2$.

Proof Suppose that $p \in \{p_1, \ldots ,p_r\}$. Since γ_Ω extends to a meromorphic function on M, there exists a neighbourhood U of p such that $T'\mathbb{P}_1$ trivializes over $V = \gamma_\Omega(U)$. So, we have

$$t : T'\mathbb{P}_1|_V \xrightarrow{\sim} V \times C$$

and

$$t \circ sp(\Omega) = (\gamma_\Omega, F) : U\setminus\{p\} \longrightarrow V \times C \ .$$

Clearly, it is sufficient to show that F does not have an essential singularity at p. Supposing that t is the restriction of a projective trivialization then we have from 3.11 that :

$$F = \frac{1}{2}(\gamma_\Omega^2 - 1)\Omega_1 - \frac{i}{2}(\gamma_\Omega^2 + 1)\Omega_2 - \gamma_\Omega\Omega_3 \ ,$$

and therefore, since Ω and hence γ_Ω are meromorphic on M, F extends over U to a map $U \longrightarrow \mathbb{P}_1$.

Remark 4.4 Note that $sp(\Omega)(p)$ does not necessarily lie on E_∞ ; in fact, $sp(\Omega) \cap E_\infty$ can be empty. E.g. take a smooth, compact algebraic curve $S \subset T'\mathbb{P}_1$ (which is not a section). Osculating S produces a null meromorphic curve Ω_S in C^3. Clearly, osculation fails at a finite number of points on S, $E \subset B_{\gamma_\Omega}$. Of course $\overline{sp(\Omega_S)(S \setminus E)} = S$.

For completeness, we state :

Theorem 4.5 Suppose that $\psi : M \longrightarrow \mathbb{P}(T'\mathbb{P}_1 \oplus \underline{C})$ is a holomorphic map of a compact Riemann surface. Then there exists $\{p_1, \ldots ,p_r\}$ such that $os(\psi) : M \setminus \{p_1, \ldots ,p_r\} \longrightarrow C^3$ is a null meromorphic curve.

Remark 4.6 Observe that osculation on 'abstract' S_2 may be viewed as follows : choose a curve E_0 from the linear system of curves of self-intersection number 2 and let C be a generic fibre, i.e. $C \cdot E_0 = 1$ and $C \cdot C = 0$. For an irreducible curve $S \subset S_2$, which is not a fibre, define $\Omega_S : S^* \longrightarrow |E_0|$ to be that curve such that :

(a) $\Omega_S(p)$ passes through p and

(b) $\partial\Omega_S(p) \cap E_0 = C \cap E_0$, where C is the fibre of S_2 that passes through p and S^* denotes that generic subset of S where such an element of $|E_0|$ exists.

The geometry of the spectral curve of $\Omega : M' \longrightarrow C^3$, intimately reflects the structure of the image $\Omega(M')$: recall from 2.19 and Definition 2.20 that $\Omega = w_{sp(\Omega)} \circ sp(\Omega)$ on M^*, where $w_{sp(\Omega)}$ arises canonically out of the geometry of $sp(\Omega)(M')$ on $T'\mathbb{P}_1$. In particular, observe that $sp(\Omega)(M')$ is the natural parameter domain for $\Omega(M')$, but may of course be singular. We shall now see that this difficulty is easily overcome if Ω is null meromorphic.

Since S_2 is an algebraic surface, it follows from the G.A.G.A. principle, see [S], that if $f : M \longrightarrow S_2$ is a holomorphic mapping of a compact Riemann surface then $f(M)$ is an algebraic curve on S_2. Now, suppose that Σ is the singular locus of $f(M)$; since $M \setminus f^{-1}(\Sigma)$ is connected, it follows that $f(M) \setminus \Sigma$ is connected and hence $f(M)$ is irreducible. An irreducible curve on an algebraic surface has a canonically associated compact Riemann surface, i.e. we have the following result, for a proof see [G&H : §4.1].

<u>Theorem 4.7</u> If $C \subset S$ is an irreducible curve on an algebraic surface then there exists a compact Riemann surface $\tilde{C}$, and a holomorphic map $\delta : \tilde{C} \longrightarrow C$ that is one-to-one over smooth points of C.

The Riemann surface $\tilde{C}$, together with the map δ is called the <u>desingularization</u> of C. We will show now that any null meromorphic curve factors through the desingularization of its spectral curve.

Theorem 4.8 Suppose that $\Omega : M\setminus\{p_1, \dots, p_r\} \longrightarrow C^3$ is a null meromorphic curve and let $S_\Omega = sp(\Omega)(M)$. There exists a holomorphic map $\Gamma : M \longrightarrow \tilde{S}_\Omega$ such that the following commutes :

$$\begin{tikzcd}
 & \tilde{S}_\Omega\setminus\Gamma\{p_1, \dots, p_r\} \arrow[d, "os(\delta)"] \\
M\setminus\{p_1, \dots, p_r\} \arrow[ur, "\Gamma"] \arrow[r, "\Omega"'] & C^3
\end{tikzcd} .$$

Proof From 4.3, S_Ω is an irreducible curve on S_2 and consequently possesses a desingularization $\tilde{S}_\Omega$. Let Σ_Ω be the singular locus of S_Ω and let $S^*_\Omega = S_\Omega\setminus\Sigma_\Omega$ and $\tilde{S}^*_\Omega = \delta^{-1}S^*_\Omega$. Since δ is 1-1 on $\tilde{S}^*_\Omega$ we have

$$\begin{tikzcd}
 & \tilde{S}^*_\Omega \arrow[d, "\delta"] \\
M\setminus sp^{-1}(\Sigma_\Omega) \arrow[ur, "\Gamma"] \arrow[r, "sp"'] & S^*_\Omega
\end{tikzcd}$$

where $\Gamma = \delta^{-1} \circ sp$. Now, at $q \in sp^{-1}(\Sigma_\Omega)$, either Γ has an essential singularity or is of the form $\zeta \longmapsto \zeta^n$ for some $n \in \mathbb{N}$. But $\delta \circ \Gamma = sp$, which extends over q (and is a rational map), thus, since δ is non-constant, this implies that Γ cannot possess an essential singularity at q and hence extends over $sp^{-1}(\Sigma_\Omega)$ to give a holomorphic map $M \longrightarrow \tilde{S}_\Omega$. Consequently we may factor $sp(\Omega)$ through $\tilde{S}_\Omega$: $sp(\Omega) = \delta \circ \Gamma$ and hence $\Omega = os(\delta) \circ \Gamma$.

Remark 4.9 This result suggests that the appropriate 'geometric data' to associate to $X = \Omega(M\setminus\{p_1, \ldots ,p_r\})$ comes from $(\tilde{S}_\Omega, os(\delta))$, i.e. we should define

$$\begin{aligned} g(X) &= g(\tilde{S}_\Omega \setminus \Gamma\{p_1, \ldots ,p_r\}) \ , \\ C(X) &= C_\delta(\tilde{S}_\Omega \setminus \Gamma\{p_1, \ldots ,p_r\}) \ , \\ \text{number of ends} &= |\Gamma\{p_1, \ldots ,p_r\}| \ , \text{ and} \\ \text{branching order} &= \beta_{os(\delta)} \ . \end{aligned}$$

This removes the flabbiness from Ossermann's inequality in 1.31 : the original data associated to $(M\setminus\{p_1, \ldots ,p_r\},\Omega)$ may be very artificial, (of course this problem does not arise for immersed surfaces).

Again, this suggests that the correct way to describe moduli spaces of null meromorphic curves in C^3 (and hence the corresponding spaces of branched minimal surfaces in $\mathbb{R}^3$) is not to fix a parameter domain M and consider $\{\Omega : M \longrightarrow C^3 \text{ ; null meromorphic}\}$, but rather to consider classes of irreducible curves on S_2. Initially one might choose curves lying in a fixed linear system on S_2. Recall that in §4.B we saw that such a curve S, will satisfy $S \sim aE_0 + bC$ for some $a,b \in Z$, so the first step is to understand the relevance of the integers $S \cdot E_0$ and $S \cdot C$ in terms of the geometry of the null meromorphic curves Ω_S. $S \cdot C$ is easy to interpret in terms of the total curvature :

<u>Proposition 4.10</u> Suppose that S is an irreducible curve on S_2 and that $\Omega_S : \tilde{S}\setminus\{p_1, \ldots, p_r\} \longrightarrow C^3$ is the associated null meromorphic curve. Then

$$C_\delta(\tilde{S}\setminus\{p_1, \ldots, p_r\}) = -4\pi S \cdot C \ .$$

<u>Proof</u> The Gauss map $\gamma_\delta = \pi|_S \circ \delta$ and hence, since δ is 1-1 off a finite set, $\deg(\gamma_\delta) = \deg(\pi|_S) = S \cdot C$. Thus, the result follows from 1.27.

The significance of $S \cdot E_0$ is not so clear. For

$$p \in S \cap E_0 \quad \text{iff} \quad \Omega_S(p) + (\gamma_\Omega(p))_0 = (\gamma_\Omega(p))_0 \quad ,$$

which occurs iff the null line $\Omega_S(p) + [\Omega_S'(p)]$ passes through 0. Thus $S \cdot E_0$ measures the number of times, counted with multiplicity, that the null lines tangent to Ω_S pass through the origin in C^3.

<u>Remark 4.11</u> Of central interest here is to determine the significance of the intersection formula :

$$E_\infty \cdot S = E_0 \cdot S - 2C \cdot S \ .$$

Rewriting this as

$$C_\delta(\tilde{S}\setminus\{p_1, \ldots, p_r\}) = 2\pi(E_\infty - E_0) \cdot S$$

we see some resemblance to Ossermann's inequality.

Observe that $E_\infty \cap S$ accounts for part of the total number of ends of Ω_S, but the significance of the algebraic intersection number is not clear.

Example 4.12 (i) Suppose that $\Omega : M\backslash\{p_1, \ldots, p_r\} \longrightarrow C^3$ is a null meromorphic curve such that $-C_\Omega(M\backslash\{p_1, \ldots, p_r\}) = 4\pi$. Since $\deg(\gamma_\Omega) = 1$, $M \simeq \mathbb{P}_1$ and $sp(\Omega) \circ \gamma_\Omega^{-1}$ is a meromorphic section of $T'\mathbb{P}_1$. We obtain all such image surfaces in C^3 by osculating meromorphic sections. Consequently we may identify the moduli space of such curves with the space of meromorphic sections of $T'\mathbb{P}_1$. This decomposes naturally into families of curves indexed by $e = E_\infty \cdot E_\sigma$, and furthermore we have $E_0 \cdot E_\sigma = e + 2$. Since γ_Ω is 1-1, the only ends on these surfaces arise where the spectral curve intersects E_∞; the intersection formula, may therefore, be easier to understand in this case.

(ii) Further restricting Ω to have 1 end, observe that we can identify the collection of null meromorphic curves with total curvature -4π and 1 end with $C[\zeta]\backslash\{\text{quadratic polynomials}\}$.

(D) Algebraic Minimal Surfaces.

In this section we briefly consider the geometry of branched minimal surfaces which arise as the real parts of null meromorphic curves. In particular we shall look at some cases in which this condition may be completely understood in terms of the geometry of ϕ in $\mathbb{R}^3$.

Definition 4.13 Suppose that $\Omega : M\setminus\{p_1, \ldots, p_r\} \longrightarrow C^3$ is a null meromorphic curve. Then $\phi = \mathrm{Re}(\Omega)$ is an algebraic minimal surface.

In 1.26 we saw that if $\phi : M' \longrightarrow \mathbb{R}^3$ is a complete, branched minimal immersion with finite total curvature such that $|B_\phi| < \infty$ then $M' \sim M\setminus\{p_1, \ldots, p_r\}$, where M is compact, and γ_ϕ extends to a holomorphic map $M \longrightarrow \mathbb{P}_1$. Consequently, $\alpha = \frac{d\phi}{d\xi}d\xi$ is a C^3-valued meromorphic 1-form on M, the real part of whose periods are all zero. In order for ϕ to be algebraic, we simply require that α is exact. Clearly, α is exact iff α_j, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, are differentials of the 2nd kind on M which have no periods. We can interpret the former condition in terms of the geometry of ϕ as a growth condition on the ends :

Suppose that ζ is a local coordinate centered at the end p, then there exists $n \in \mathbb{N}$ such that

$$\frac{d\phi}{d\xi}(\xi) = a_{-n}\xi^{-n} + \ldots a_{-1}\xi^{-1} + a_0 + a_1\xi + \ldots$$

where $a_j \in C^3$, and hence

$$\phi(\xi) = \mathrm{Re}\left\{\frac{a_{-n}}{-n+1}\,\xi^{-n+1} + \ldots + a_{-1}\log(\xi) + a_0\xi + \ldots\right\} + C\ .$$

Since ϕ is single-valued observe that $a_{-1} \in \mathbb{R}^3$.

Definition 4.14 $|a_{-1}|$ is the modulus of logarithmic growth of the end. In particular, if $a_{-1} = 0$ then the end is said to have zero logarithmic growth.

Remark 4.15 If the end p is embedded and has zero logarithmic growth then there exists coordinates on $\mathbb{R}^3$ and $\xi = \xi_1 + i\xi_2$ centered at p such that ϕ has the following form near p :

$$\phi(\xi) = \begin{pmatrix} \xi_1 |\xi|^{-2} + \mathrm{Re}(\xi h_1) \\ \xi_2 |\xi|^{-2} + \mathrm{Re}(\xi h_2) \\ \mathrm{Re}(\xi h_3) \end{pmatrix}$$

where h_j are holomorphic functions. This follows easily from the fact that the end is embedded iff α has a pole of order 2 at p. (We are assuming here that M is conformally equivalent to $\tilde{S}_\phi$.) For further details see [Sc] and [Br]. For 'genus zero' we can give the following geometric criteria :

Proposition 4.15 A branched minimal immersion $\phi : \mathbb{P}_1 \setminus \{p_1, \dots, p_r\} \longrightarrow \mathbb{R}^3$ is algebraic iff ϕ is complete, has finite total Gaussian curvature, $|B_\phi| < \infty$ and the ends have zero logarithmic growth.

For $r = 1$, the residue theorem gives the following

Corollary 4.16 (i) A branched minimal immersion $\phi : C \longrightarrow \mathbb{R}^3$ is algebraic iff ϕ is complete, $|B_\phi| < \infty$ and $C_\phi(C) > -\infty$.

(ii) If furthermore, $C_\phi(C) = -4\pi$, then ϕ arises as the real part of a null meromorphic curve obtained by osculating a meromorphic section of $T'\mathbb{P}_1$. Thus the moduli space of such minimal surfaces is

$$(\mathbb{C}[\zeta]\backslash\{\text{quadratic polynomials}\})/\mathbb{R}^3$$

where the equivalence relation is given by :

$$p(\zeta) \sim p(\zeta) + i\left\{\left(\frac{x_1 + ix_2}{2}\right) + x_3\zeta - \left(\frac{x_1 - ix_2}{2}\right)\zeta^2\right\}$$

for $(x_1, x_2, x_3) \in \mathbb{R}^3$. (This is simply because Ω and $\Omega + ix$ project to the same surface in $\mathbb{R}^3$.)

We close this section with a result of Meek's, see Lemma 1 of [M], which provides another source of algebraic minimal surfaces.

<u>Proposition 4.17</u> Let $\pi : \mathbb{P}_1 \longrightarrow \mathbb{RP}_2$ be the usual double covering and suppose that $\pi\{p_1, \dots, p_{2r}\} = \{q_1, \dots, q_r\}$. If $\phi : \mathbb{P}_1\backslash\{p_1, \dots, p_{2r}\} \longrightarrow \mathbb{R}^3$ is a branched minimal surface with an algebraic Gauss map and there exists $\tilde{\phi} : \mathbb{RP}_2\backslash\{q_1, \dots, q_r\} \longrightarrow \mathbb{R}^3$ such that $\phi = \tilde{\phi} \circ \pi$, then ϕ is algebraic.

<u>(E) Ends</u>

This section consists of some preliminary observations regarding the relationship between the end structure of an algebraic minimal surface and the structure of the branch points on its spectral curve, viewed as a branched covering of $\mathbb{P}_1$.

Recall that ends occur on Ω where $sp(\Omega)$ either intersects E_∞ or fails to be transverse to a fibre. Of course, not all of the latter give rise to ends, since they may simply correspond to zeros of Gaussian curvature, i.e. branch points of the Gauss map. (For a beautiful example of a branch point of the Gauss map see the picture of the Jorge - Meeks Trinoid in [B&C]. Every plane which contains the normal vector to $\phi(0)$ intersects the surface in a cubic curve with the point of inflection at $\phi(0)$, giving $g_\phi(z) = z^2$.) Now, recall that if $\phi : M\backslash\{p_1, \ldots ,p_r\} \longrightarrow \mathbb{R}^3$ is an immersion then an end p is embedded iff $\frac{d\phi}{d\zeta}$ has a pole of order 2 at p. Clearly, if ϕ is not ramified, i.e. M is conformally equivalent to $\tilde{S}_\phi$, this is still a necessary condition for embeddedness of p. Consequently, we have :

<u>Proposition 4.18</u> If $\phi : \mathbb{P}_1\backslash\{p_1, \ldots ,p_r\} \longrightarrow \mathbb{R}^3$ is algebraic with total curvature -4π then none of the ends can be embedded.

<u>Proof</u> Recall that all such image surfaces arise by osculating meromorphic sections of $T'\mathbb{P}_1$ and that the ends of Ω_f occur at the poles of f. But, if $f(\zeta) \sim \zeta^{-n}$, the formulae of 3.11 imply that $\Omega_1, \Omega_2 \sim \zeta^{-(n+2)}$, where $\phi = \mathrm{Re}(\Omega)$, and hence that $\frac{d\phi}{d\zeta} \sim \zeta^{-(n+3)}$. Since $n \geqslant 1$, the end cannot be embedded.

<u>Corollary 4.19</u> The catenoid can be characterised as the unique complete branched minimal surface in $\mathbb{R}^3$ with $|B_\phi|$ finite and total curvature -4π, which has embedded ends.

The behaviour of the following class of simple examples is quite instructive :

Example 4.20 Consider the rational curve $S_{p,q}$ on $S(T'\mathbb{P}_1 \oplus \underline{C})$ that is given over a neighbourhood of 0, by the equation $\eta^q = \zeta^p$, where ζ is a l.a.c. on $\mathbb{P}$ and p and q are coprime integers. Observe that $S_{p,q}$ is described over a neighbourhood of infinity by the equation $\mu^q = w^{2q-p}$, where $w = \zeta^{-1}$ (and consequently that osculating $S_{2q-p,q}$ gives essentially the same null curve in C^3 as osculation of $S_{p,q}$). We view $S_{p,q}$ as the completion of the curve in C^2 given by $u \longmapsto (u^q, u^p)$ and use this parameterization to describe the null meromorphic curve $\Omega^{p,q}$ in C^3, obtained by osculating $S_{p,q}$. First, observe that $S_{p,q}$ is transverse to the fibres over $\mathbb{P}_1$ except at $\zeta = 0,\infty$, and hence we can focus attention on these 2 branch points. In particular we can see immediately that $\Omega^{p,q}$ has at most 2 ends. We can describe $S_{p,q}$ over a neighbourhood of 0 by a function of the form :

$$f(\zeta) = \zeta^{p/q} \quad , \text{ and hence}$$

$$f'(\zeta) = \frac{p}{q}\zeta^{(\frac{p}{q}-1)} \quad \text{and} \quad f''(\zeta) = \frac{p}{q}\left(\frac{p}{q}-1\right)\zeta^{(\frac{p}{q}-2)} \quad .$$

Consequently since $\zeta = u^q$, one can easily check that the Weierstrass formulae in free form yield :

$$\Omega_1^{p,q}(u) = \frac{p}{2q}\left(\frac{p}{q}-1\right) u^{p-2q} - \left\{1 - \frac{3p}{2q} + \frac{p^2}{2q^2}\right\} u^p$$

$$\Omega_2^{p,q}(u) = \frac{ip}{2q}\left(\frac{p}{q}-1\right) u^{p-2q} + i\left\{1 - \frac{3p}{2q} + \frac{p^2}{2q^2}\right\} u^p$$

$$\Omega_3^{p,q}(u) = \frac{p}{q}\left(\frac{p}{q} - 2\right) u^{p-q} \quad .$$

$\phi^{p,q} = \mathrm{Re}(\Omega^{p,q})$ gives an algebraic minimal surface in $\mathbb{R}^3$ with total curvature $-4\pi q$.

For example, if p=3 and q=1 then we get Enneper's surface by taking the real part. If p=3, q=2 we get :

$$\Omega_1^{3,2}(u) = \frac{1}{8}(3u^{-1} + u^3)$$

$$\Omega_2^{3,2}(u) = \frac{i}{8}(3u^{-1} - u^3)$$

$$\Omega_3^{3,2}(u) = -\frac{3}{4}u \quad .$$

Taking real part we obtain :

$$\phi_1^{3,2}(u) = \frac{1}{8}\left\{\frac{3x}{x^2+y^2} + x^3 - 3xy^2\right\}$$

$$\phi_2^{3,2}(u) = \frac{1}{8}\left\{\frac{3y}{x^2+y^2} - y^3 + 3yx^2\right\}$$

$$\phi_3^{3,2}(u) = -\frac{3}{4}x \quad .$$

<u>Remark 4.21</u> (i) If $\frac{p}{q} > 2$ then $\Omega(0)$ is finite. (So we have a branch point of the Gauss map.) This gives a branch point of $\Omega^{p,q}$ (and hence of $\phi^{p,q}$) : for $\frac{d}{du}\Omega_3^{p,q}(u) = 0$ iff u=0 and if $p > 2q$ then $\frac{d}{du}\Omega_1^{p,q}(0) = \frac{d}{du}\Omega_2^{p,q}(0) = 0$. Also note that $p > 2q$ forces $S_{p,q}$ to intersect E_∞ over infinity on $\mathbb{P}_1$, for $p > 2q$ iff $\mu^q = w^{2q-p}$ has a pole at 0.

(ii) $0 < p/q < 2$ forces ends at $\{0,\infty\}$ and hence the surface is immersed for these values.

(iii) If $1 < p/q < 2$ then $\Omega_1^{p,q}(0) = \Omega_2^{p,q}(0) = \infty$ but $\Omega_3^{p,q}(0)$ is finite. Consequently the end is asymptotic to a plane. (However, note that this does not imply that the surface is contained in a $\frac{1}{2}$-space – as sketches of cross-sections of $\Omega^{3,2}$ reveal.) $\Omega^{p,q}$ cannot have both ends of this type for $1 < p/q < 2$ iff $0 < 2-p/q < 1$.

(iv) For $p/q < 1$, all $\Omega_j^{p,q}(0)$ are infinite. It would be interesting to understand the difference between $0 < p/q < 1$ and $p/q < 0$ in terms of the geometry of $\Omega^{p,q}$.

More generally, for an irreducible curve $S \subset \mathbb{P}(T'\mathbb{P} \oplus \underline{C})$ with desingularization $\tilde{S} \xrightarrow{\delta} S$ we make the following simple observations:

View S as a branched covering of $\mathbb{P}_1$ and recall that, with respect to a projective trivialization we can write $\delta(\xi) = (\gamma_{os(\delta)}, F)$, where F is a meromorphic function. At a branch point p of $\gamma_{os(\delta)}$, choosing coordinates so that $\gamma_{os(\delta)}(\xi) = \xi^q = \zeta$ we have the Puiseux series representation of $\delta(U)$, for U a neighbourhood of p :

$$f(\zeta) = \sum_{-\infty}^{\infty} a_r \zeta^{r/q} .$$

Of course, $a_r = 0$ for all $r \leq n$, for some $n \in Z$. The behaviour of Ω_S near p is determined by n/q, e.g.

$n/q > 2$ implies that $\Omega_S(p)$ is finite.

$1 < n/q < 2$ implies that $\Omega_{S,3}(p)$ is finite.

$n/q < 1$ implies that $\Omega_{S,j}(p) = \infty$ for $j = 1,2,3$.

(F) Yang - Mills - Higgs SU(2) - Monopoles.

In this section we return to Hitchin's paper [H2], and in particular, his observation that one may associate a minimal surface to an SU(2) - monopole We describe the essential features of the global structure of this surface and observe that it reflects the soliton nature of its origin. Finally we consider some speculation regarding the existance of soliton families of minimal surfaces. We begin by describing, very briefly, some elements of monopole theory : for further details see [A&H], [D], [H2], [H3] and [J&T].

Suppose ∇, a connection on a principal SU(2) - bundle over $\mathbb{R}^3$, and Φ, a section of the associated bundle of Lie algebras, are coupled by the Bogomolny equation :

$$\nabla\Phi = *F$$

(here F is the curvature of ∇, and * is the Hodge *-operator). Then (∇,Φ) is an absolute minimum of the energy functional

$$E(\nabla,\Phi) = \int_{\mathbb{R}^3} |F|^2 + |\nabla\Phi|^2 \, dx$$

within the fixed topological type, and (∇,Φ) is a static Yang-Mills-Higgs SU(2)-monopole in the Prasad-Sommerfield limit. Finiteness of the energy forces the boundary condition 'at infinity' that Φ asymptotically approaches a fixed adjoint orbit in $su(2) \simeq \mathbb{R}^3$, in fact more is true for we have

$$\|\Phi\| = 1 - \frac{k}{r} - O(r^{-2})$$

$$\frac{\partial\|\Phi\|}{\partial\Omega} = O(r^{-2})$$

$$\|\nabla\Phi\| = O(r^{-2}) \qquad \text{as} \quad r \longrightarrow \infty$$

where $\|\Phi\|^2 = -\frac{1}{2}\mathrm{Trace}(\Phi^2)$ and $\frac{\partial}{\partial\Omega}$ denotes the angular derivative, see [J&T]. Now, Hitchin has shown in [H2], that a solution (∇,Φ) of the Bogomolny equation, subject to these boundary conditions, may be faithfully encoded into the <u>spectral curve</u> of the monopole, which is an algebraic curve on $T'\mathbb{P}_1$ constructed in the following way. Given (∇,Φ), on each line ℓ in $\mathbb{R}^3$, we have the ordinary differential equation

$$(\nabla_\ell - i\Phi)\psi = 0$$

where ∇_ℓ denotes the covariant derivative in the direction of ℓ and ψ is a 2-component function on ℓ. Then, viewing $T'\mathbb{P}_1$ as the space of oriented lines in $\mathbb{R}^3$, the subset of lines $\ell \in T'\mathbb{P}_1$ for which the above equation has a square-integrable solution, is a compact algebraic curve, from which the monopole can be reconstructed. In particular, the topological charge $k \in \mathbb{N}$ of the monopole, which is the degree of the map $S^2 \longrightarrow S^2$ given by restricting Φ to a large sphere in $\mathbb{R}^3$, may be read off the spectral curve as $S \cdot C$, where C is a generic fibre of $T'\mathbb{P}_1 \longrightarrow \mathbb{P}_1$. Consequently the next result, which gives an integral representation formula for the charge of a monopole, follows immediately from 4.10.

Theorem 4.22 Suppose that $S \subset T'\mathbb{P}_1$ is the spectral curve of an SU(2)-monopole of charge k and that $\tilde{S} \xrightarrow{\delta} S$ is its desingularization. There exists a finite set of points $\{p_1, \ldots, p_r\} \subset \tilde{S}$ such that osculation of S induces a semi-metric g on $\tilde{S}\setminus\{p_1, \ldots p_r\}$ whose Gaussian curvature K_g satisfies :

$$k = -\frac{1}{4\pi} \int_{\tilde{S}\setminus\{p_1, \ldots, p_r\}} K_g \, dA_g$$

Corollary 4.23 The algebraic minimal surface $\phi = \mathrm{Re}(\mathrm{os}(\delta))$ has total curvature equal to -4π x (charge of monopole) .

Remark 4.24 (i) Observe that this formula is not a consequence of the Gauss - Bonnet formula on S. For example if g=1 then S is smooth and K=2, see [H2] ; but of course any metric on S has total curvature equal to zero.

(ii) The points $\{p_1, \ldots, p_r\}$ arise intrinsically and correspond to the ends of ϕ. Recall that in §1.F we saw that $\phi(\tilde{S}\setminus\{p_1, \ldots, p_r\})$, viewed from infinity, looks like a finite collection of planes intersecting at the origin : $\pi \circ \delta(p_1), \ldots, \pi \circ \delta(p_r)$ are the normal vectors to these 'planes'.

(iii) The minimal surface associated to a static monopole has some aspects of soliton structure : for, observe that the Gaussian curvature is highly localized on the image surface in $\mathbb{R}^3$.

Consider the family of algebraic minimal surfaces obtained by osculating the family of spectral curves associated to a geodesic on the moduli space of SU(2)-monopoles of charge k, (the low-energy approximation to monopole motion, see [A&H]). This family should exhibit interesting soliton behaviour. For k=2, the spectral curve is either smooth and elliptic or consists of a pair of sections. In the elliptic case they are completions in $T'\mathbb{P}_1$ of cubic curves with the normal form :

$$\eta^2 = r_1\zeta^3 - r_2\zeta^2 - r_1\zeta \quad , \quad r_i \in \mathbb{R}(\geqslant 0) .$$

Consequently such an S is an embedding in $T'\mathbb{P}_1$ of C modulo a lattice, via a Weierstrass $\wp$-function and its derivative : $\eta \sim \wp'$, $\zeta \sim \wp$. Thus, osculation gives algebraic minimal surfaces in $\mathbb{R}^3$ whose Gauss maps are Weierstrass $\wp$-functions. Furthermore, given the explicit representation of S, it should be easy to calculate Ω_S.

Osculating the family of elliptic curves associated to a 'slowly-moving' SU(2)-monopole of charge 2 appears to lead to an interesting bifurcation phenomenon. Asymptotically, this family breaks up into a pair of sections (corresponding to the fact that well-separated monopoles are particle-like), see [Hurl] and [A&H : §7]. Consequently, the associated family of minimal surfaces degenerates into a pair of points on the 2-sphere at infinity. This seems to accord well with our knowledge of the global structure of complete branched minimal surfaces with finite branching and finite total curvature, see §1.F. For, it seems likely that as the elliptic curves approximate to 2 sections, the Gaussian curvature on the associated minimal surfaces concentrates in 2 regions which are moving apart, (recall that osculating a section gives a point). Thus the picture 'from infinity' forces some sort of degeneration to occur, because the curvature must be 'concentrated in a finite region' for the surface to look like a finite collection of planes through the origin. One can obviously view these concentrations of curvature as 'particles' and study their 'interactions'.

Problems 4.25 (1) Construct the minimal surface directly from the original data (∇, Φ). In particular, write down the Gaussian curvature and the integral representation formula of 4.22 in terms of the original data.

(2) Is there a physical interpretation of Ω_S or ϕ_S? Note that the 'curvature field' of ϕ_S is asymptotically zero. Is there some relationship with the work of Hurtubise on the asymptotic Higgs field, see [Hur2] ?

(3) Interpret the constraints on a spectral curve S in terms of Ω_S - see 7.3 of [H2]. ((iv)is simple.)

(4) Explain the significance of those points $\{p_1, \ldots, p_r\}$ where osculation fails on S.

(5) Formulate a sensible way to measure the degree of concentration of Gaussian curvature.

(6) Show that as an elliptic curve approximates to 2 sections the Gaussian curvature concentrates in 2 regions. (This should involve considering the fact that the distribution of tangent affine null planes approximates that of a pair of points.)

Appendix A : Branch Points and Ramification.

Let X be an n-dimensional complex manifold and suppose that $\mu : M \longrightarrow X$ is a holomorphic curve. Furthermore suppose that ζ is a local coordinate on a neighbourhood of $\zeta_0 \in M$, and that $\chi : V \longrightarrow C^n$ is a chart centred at $\mu(\zeta_0)$, where V is a neighbourhood of $\mu(\zeta_0)$.

Definition 1 A curve $\mu : M \longrightarrow X$, has a branch point of order m at ζ_0, if the Taylor expansion of $\mu_\chi = \chi \circ \mu$ has the following form in the coordinate ζ :

$$\mu_\chi(\zeta) = a + \zeta^{m+1}\tilde{\mu}(\zeta) ,$$

where $a \in C^n$ and $\tilde{\mu}(0) \neq 0$. (Equivalently, $\mu_\chi^{(1)}(0) = \ldots = \mu_\chi^{(m)}(0) = 0$ and $\mu_\chi^{(m+1)}(0) \neq 0$.) Let $\underline{br_\zeta(\mu)}$ denote the order of branching of μ at ζ. The following simple lemmas show that this notion is well-defined.

Lemma 2 Suppose that $\chi_1, \chi_2 : V \longrightarrow C^n$ are charts centred at $\mu(\zeta_0)$ and let $\mu_1 = \chi_1 \circ \mu$ and $\mu_2 = \chi_2 \circ \mu$. Then

$$\mu_1^{(1)}(0) = \ldots = \mu_1^{(m)}(0) = 0 \quad \text{and} \quad \mu_1^{(m+1)}(0) \neq 0$$

iff

$$\mu_2^{(1)}(0) = \ldots = \mu_2^{(m)}(0) = 0 \quad \text{and} \quad \mu_2^{(m+1)}(0) \neq 0 .$$

Proof Since $\mu_2 = \Theta \circ \mu_1$, where $\Theta = \chi_2 \circ \chi_1^{-1}$,

$$\mu_2^{(1)} = (\Theta_{ij} \circ \mu_1)\mu_1^{(1)} , \quad \text{where} \quad (\Theta_{ij}) = \partial\Theta .$$

From Leibnitz's formula, each term in the i-th component of the r-th derivative, $\mu_{2,i}^{(r)}$, involves a component of $\mu_1^{(\ell)}$, for some $1 < \ell < r$; thus since $\mu_1^{(\ell)}(0) = 0$ for $1 \leqslant \ell \leqslant r$, observe that if $r \leqslant m$ then $\mu_2^{(r)}(0) = 0$. For $i = 1, \ldots ,n$ we have

$$\mu_{2,i}^{(m+1)} = \text{terms involving } \mu_{1,d}^{(\ell)} + \sum_j (\Theta_{ij} \circ \mu_1)\mu_{1,j}^{(m+1)}$$

where $1 \leqslant \ell \leqslant m$, $1 \leqslant d \leqslant n$, thus

$$\mu_2^{(m+1)}(0) = (\Theta_{ij} \circ \mu_1)\mu_1^{(m+1)}(0) .$$

Since Θ is a biholomorphism, $\mu_1^{(m+1)}(0) \neq 0$ implies that $\mu_2^{(m+1)}(0) \neq 0$. $\mu_1 = \Theta^{-1} \circ \mu_2$ reverses the arguement.

Lemma 3 Suppose that $\mu : M \longrightarrow C^n$ is holomorphic and that ζ and ω are local holomorphic coordinates at $p \in M$. Then

$$\frac{d\mu}{d\zeta}(p) = \ldots = \frac{d^m\mu}{d\zeta^m}(p) = 0 \quad \text{and} \quad \frac{d^{m+1}}{d\zeta^{m+1}}\mu(p) \neq 0$$

iff

$$\frac{d\mu}{d\omega}(p) = \ldots = \frac{d^m\mu}{d\omega^m}(p) = 0 \quad \text{and} \quad \frac{d^{m+1}}{d\omega^{m+1}}\mu(p) \neq 0 .$$

The proof follows easily from repeated differentation of $\frac{d\mu}{d\omega} = \frac{d\zeta}{d\omega}\frac{d\mu}{d\zeta}$.

The following notion which is derived from [G-O-R], measures the extent to which the branching order of a holomorphic curve at a point is implicit in the parameterization.

<u>Definition 4</u> A curve $\mu : M \longrightarrow X$ has <u>ramification index r at ζ_0</u> if r is the largest positive integer for which there exists a holomorphic map $\rho : U \longrightarrow D$, where U is a neighbourhood of ζ_0 and D is a domain in C, such that

(i) $\rho : U \setminus \{\zeta_0\} \longrightarrow D \setminus \{\rho(\zeta_0)\}$ is an $(r+1)$-sheeted cover.

(ii) There exists a holomorphic map $\eta : D \longrightarrow X$ such that $\mu = \eta \circ \rho$.

Let $\underline{\mathrm{ram}_{\zeta_0}(\mu)}$ denote the ramification index of μ at ζ_0.

<u>Remark 5</u> If $\mathrm{br}_{\zeta_0}(\mu) = \mathrm{ram}_{\zeta_0}(\mu)$ then μ is <u>totally ramified</u> at ζ_0 which is then referred to as a <u>false branch point</u>.

<u>Proposition 6</u> Let M_1 and M_2 be Riemann surfaces and suppose that $\rho : M_1 \longrightarrow M_2$ is a holomorphic map and $\eta : M_2 \longrightarrow X$ is a holomorphic curve. Then for $\mu = \eta \circ \rho$ we have

$$\mathrm{br}_{\zeta_0}(\mu) = (\mathrm{br}_{\zeta_0}(\rho) + 1)(\mathrm{br}_{\rho(\zeta_0)}(\eta) + 1) - 1 .$$

<u>Proof</u> There exist charts ψ_1 and ψ_2 centred at ζ_0 and $\rho(\zeta_0)$ respectively such that

$$\psi_2 \circ \rho \circ \psi_1^{-1}(\zeta) = \zeta^{r+1}$$

where $r = br_{\zeta_0}(\rho)$. If $\chi : V \longrightarrow C^n$ is a chart centred at $\mu(\zeta_0)$ then

$$\chi \circ \eta \circ \psi_2^{-1}(\xi) = a + \xi^{k+1}\tilde{\eta}(\xi)$$

where $\tilde{\eta}(0) \neq 0$ and $k = br_{\rho(\zeta_0)}(\eta)$. Consequently we have

$$\chi \circ \eta \circ \rho \circ \psi_1^{-1}(\zeta) = a + \tilde{\eta}(\zeta^{r+1})\zeta^{(r+1)(k+1)}$$

and hence $br_{\zeta_0}(\mu) = (r+1)(k+1) - 1$.

<u>Corollary 7</u> For a holomorphic curve $\mu : M \longrightarrow X$ and $\zeta_0 \in M$, the integer

$$Br_{\zeta_0}(\mu) = \frac{br_{\zeta_0}(\mu) + 1}{ram_{\zeta_0}(\mu) + 1} - 1$$

gives a measure of the branching of the image surface, $\mu(M)$, at $\mu(\zeta_0)$. In particular, if $Br_{\zeta_0}(\mu) = 0$ observe that $\mu(U)$ is biholomorphic to a disc for some sufficiently small neighbourhood U of ζ_0.

Appendix B : Additional Problems.

(1) Describe the Weierstrass - Hitchin correspondence for a general Einstein - Weyl space, see [H1].

(2) Determine the class of curves in SL(2,C) given by this correspondence.

(3) Develop this theory in the context of curves lying on points of the family of quadrics in $\mathbb{P}_3$ which is parameterised by the curvature c, of a 3-dimensional space form and degenerates at c=0 to a singular cone, see [A].

(4) Does this relate to Bryant's work on Willmore surfaces in S^3 ? See [Br].

(5) Understand how non-orientability manifests itself in terms of the Weierstrass - Hitchin correspondence.

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